ALGEBRAS OF TWISTED CHIRAL DIFFERENTIAL OPERATORS AND AFFINE LOCALIZATION OF $\mathfrak{g}$-MODULES

T.ARAKAWA, D.CHEBOTAROV, AND F.MALIKOV

Abstract. We propose a notion of algebra of *twisted* chiral differential operators over algebraic manifolds with vanishing 1st Pontrjagin class. We show that such algebras possess families of modules depending on infinitely many complex parameters, which we classify in terms of the corresponding algebra of twisted differential operators. If the underlying manifold is a flag manifold, our construction recovers modules over an affine Lie algebra parameterized by opers over the Langlands dual Lie algebra. The spaces of global sections of "smallest" such modules are irreducible $\hat{\mathfrak{g}}$-modules and all irreducible $\mathfrak{g}$-integrable $\hat{\mathfrak{g}}$-modules at the critical level arise in this way.

1. Introduction

1.1. **Algebras.** Algebras of twisted differential operators (TDO) were proposed by Bernstein and Beilinson [BB1, BB2] as a tool to study representation theory of simple complex Lie algebras. To give an example, consider the projective line $\mathbb{P}^1$ with an atlas consisting of 2 copies of $\mathbb{C}$ with coordinates x and y resp. so that $y = 1/x$. One has

$$\partial_y = -x^2\partial_x. \tag{1.1}$$

This defines the tangent sheaf $\mathcal{T}_{\mathbb{P}^1}$; $\mathcal{T}_{\mathbb{P}^1}$ is a Lie algebroid and its universal enveloping algebra is the algebra of differential operators $\mathcal{D}_{\mathbb{P}^1}$.

This construction is twisted by postulating the following transition function

$$\partial_y = -x^2\partial_x + \lambda x, \lambda \in \mathbb{C}. \tag{1.2}$$

The result is the algebra of twisted differential operators $\mathcal{D}^{\lambda}_{\mathbb{P}^1}$. It is isomorphic to $\mathcal{D}_{\mathbb{P}^1}$ locally, but not globally; for example, if λ is an integer, then $\mathcal{D}^{\lambda}_{\mathbb{P}^1}$ is the algebra of differential operators acting on the sheaf $\mathcal{O}(\lambda)$.

Such algebras of locally trivial twisted differential operators can be defined for an arbitrary smooth algebraic variety X; their isomorphism classes are in 1-1 correspondence with $H^1(X, \Omega^{1,cl}_X)$. Thus for each $\lambda \in H^1(X, \Omega^{1,cl}_X)$, there is an algebra $\mathcal{D}^{\lambda}_X$.

This construction can be further generalized to include algebras that are not isomorphic to $\mathcal{D}_X$ even locally. These are classified by the hypercohomology group $H^1(X, \Omega^1_X \to \Omega^{2,cl}_X)$, and we obtain a $\mathcal{D}^{\lambda}_X$ for each $\lambda \in H^1(X, \Omega^1_X \to \Omega^{2,cl}_X)$.

Introduced in [MSV, GMS1] – and in [BD1] in the language of chiral algebras – are algebras of chiral differential operators, CDO; these are sheaves of vertex algebras of a certain type that resemble algebras $\mathcal{D}_X$ in some respects. A CDO over X may or may not exist; in fact, it exists if and only if $ch_2(\mathcal{T}_X) \in H^2(X, \Omega^2_X \to \Omega^{3,cl}_X)$ equals 0. If it does, then the isomorphism classes of CDO-s over X are a

torsor over $H^1(X, \Omega_X^2 \to \Omega_X^{3,cl})$ – note that the degree has jumped in comparison with the case of twisted differential operators.

One can argue, therefore, that all CDO-s are twisted, because there is no distinguished one and, worse still, there may be none at all. Nevertheless, it is the purpose of this paper to introduce a class of *twisted chiral differential operators*, TCDO, so that all of the above CDO-s will appear untwisted.

To give a flavor of the construction, let us return to the case of $X = \mathbb{P}^1$. $\mathbb{P}^1$ carries a unique up to isomorphism CDO, $\mathcal{D}^{ch}_{\mathbb{P}^1}$; it is defined by means of the following 'chiralization' of (1.1):

$$\partial_y = -x_{(-1)}x_{(-1)}\partial_x - 2\partial(x), \tag{1.3}$$

where we have let ourselves use freely some of vertex algebra and CDO notation; for example, x and ∂_x are fields associated (in some sense) to the coordinate and derivation so denoted, and $\partial(x)$ means the canonical vertex algebra *translation operator* applied to x.

Next, one would like to find a chiral version of (1.2). Writing simply $\partial_y = -x_{(-1)}x_{(-1)}\partial_x - 2\partial(x) + \lambda x$, $\lambda \in \mathbb{C}$, is possible but uninteresting and ultimately unhelpful. It appears that the right thing to do is to chiralize not any of $\mathcal{D}^{\lambda}_X$ but their universal version, $\mathcal{D}^{tw}_X$. In the case of $\mathbb{P}^1$, this means to define $\mathcal{D}^{tw}_{\mathbb{P}^1}$ as a $\mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{C}[\lambda]$-module using the same (1.2) with λ not a number but a variable.

The chiral version of this is as follows: replace $\mathbb{C}[\lambda]$ with $H_{\mathbb{P}^1} = \mathbb{C}[\lambda, \partial(\lambda), \partial^2(\lambda), ...]$, the commutative vertex algebra of differential polynomials on $\mathbb{C}$, and then define an algebra of twisted chiral differential operators, $\mathcal{D}^{ch,tw}_{\mathbb{P}^1}$, to be the sheaf of vertex algebras locally isomorphic to $\mathcal{D}^{ch}_{\mathbb{P}^1} \otimes H_{\mathbb{P}^1}$ with the following transition functions

$$\partial_y = -x_{(-1)}x_{(-1)}\partial_x - 2\partial(x) + \lambda_{(-1)}x. \tag{1.4}$$

Similarly, we construct for an arbitrary compact smooth X the universal algebra of twisted differential operators, $\mathcal{D}^{tw}_X$; it is an algebra over $\mathbb{C}[H^1(X, \Omega^1_X \to \Omega^{2,cl}_X)]$ such that being quotiented out by the maximal ideal of a point $\lambda \in H^1(X, \Omega^1_X \to \Omega^{2,cl}_X)$ it gives $\mathcal{D}^{\lambda}_X$. We then chiralize this construction and obtain, for each CDO $\mathcal{D}^{ch}_X$, a twisted CDO $\mathcal{D}^{ch,tw}_X$, a sheaf of vertex algebras, which locally, but not globally, looks like $\mathcal{D}^{ch,tw}_X \otimes H_X$, where H_X is the algebra of differential polynomials on $H^1(X, \Omega^1_X \to \Omega^{2,cl}_X)$.

Apart from serving as a prototype, algebras of twisted differential operators are directly linked to algebras of twisted chiral differential operators via the notion of the Zhu algebra [Zhu], and this is another topic of the present paper. Zhu attached to each graded vertex algebra V an associative algebra, $\mathcal{Z}hu(V)$. We show that the sheaf associated to the presheaf $X \supset U \mapsto \mathcal{Z}hu(\mathcal{D}^{ch,tw}_X(U))$ is precisely $\mathcal{D}^{tw}_X$.

$\mathcal{Z}hu(V)$ controls representation theory of V, the subject to which we now turn.

1.2. **Modules.** Note that $\mathcal{D}^{ch,tw}_X$ is not a deformation of $\mathcal{D}^{ch}_X$, not technically at least, but it has a rich representation theory. In particular, it has families of modules that are indeed deformations of those over $\mathcal{D}^{ch}_X$, and this is why $\mathcal{D}^{ch,tw}_X$ may be of interest.

Zhu showed that under some restrictions, a V-module is the same as a $\mathcal{Z}hu(V)$-module. It follows easily that (under similar restrictions) a $\mathcal{D}^{ch}_X$-module is the same as a $\mathcal{D}_X$-module, a result that is a bit disheartening.

One of those restrictions is that a V-module be graded. In the case of $\mathcal{D}_X^{ch,tw}$ let us relax this by demanding that modules be only filtered. Now note that H_X belongs to the center of $\mathcal{D}_X^{ch,tw}$. Therefore, we can take any $\mathcal{D}_X^{ch,tw}$-module, for example one coming from a $\mathcal{D}_X^{tw}$-module, and quotient it out by a character of H_X.

It is easy to see that a character of H_X is an element of $H^1(X, \Omega_X^1 \to \Omega_X^{2,cl})((z))$. Among those a special role is played by characters with regular singularities, $\chi(z) = \chi_0/z + \chi_{-1} + \chi_{-2}z + \cdots$, $\chi_j \in H^1(X, \Omega_X^1 \to \Omega_X^{2,cl})$.

Arguing along these lines we prove that

A $\mathcal{D}_X^{ch,tw}$-module with central character $\chi(z)$ is the same as a $\mathcal{D}_X^{\chi_0}$-module if $\chi(z)$ has regular singularity and zero otherwise.

The content of this assertion is not in the vanishing result, which is valid only under some technical restrictions that we have skipped anyway, but in the explicit construction of a variety of modules labeled by characters $\chi(z)$. Here is one example that this construction generalizes.

Let $X = G/B$ be a flag manifold. Then a $\mathcal{D}_X^{\chi_0}$-module is essentially the same as a $\mathfrak{g}$-module with central character determined by χ_0. Applying the above construction to the contragredient Verma module over $\mathfrak{g}$, we obtain a sheaf whose space of sections over the big cell is the Wakimoto module over $\widehat{\mathfrak{g}}$ at the critical level quotiented out by the central character $\chi(z)$ [FF1, F1], $\chi(z)$ being interpreted in this case as an *oper* for the Langlands dual group. Of course, this is a beginning of the representation-theoretic input to the Beilinson-Drinfeld construction of Hecke eigensheaves on Bun_G, [BD2], also [F2, F3]. Therefore, what we are doing can be thought of as providing "operatic" parameters in the case of an arbitrary manifold; and indeed, the spectacular work by Feigin and Frenkel served as a major source of inspiration for us.

Furthermore, we prove that

if the $\mathfrak{g}$-module we start with is simple and finite-dimensional, then the space of global sections of the corresponding $\mathcal{D}_{G/B}^{ch,tw}$-module is irreducible and isomorphic to the Weyl module over $\widehat{\mathfrak{g}}$ at the critical level quotiented out by the central character.

The irreducibility of Weyl modules at the critical level quotiented out by the central character is a result of Frenkel and Gaitsgory, which was anticipated in [FG2] and proved in [FG3]. Our analysis of the spaces of global sections heavily relies on techniques and results [FG1, FG2, FG3].

Let us see how this (and a bit more) comes about in the case of $X = \mathbb{P}^1$.

This case is described by explicit formulas (1.2) and (1.4). If we let in (1.2) $\lambda = n \in \mathbb{Z}$, then the "smallest" $\mathcal{D}_{\mathbb{P}^1}^n$-module is $\mathcal{O}(n)$. To make our life easier, let $\chi(z) = n/z$. This is the case when the resulting $\mathcal{D}_{\mathbb{P}^1}^{ch,tw}$-module is actually graded; denote it by $\mathcal{O}(n)^{ch}$. Note that when $n = 0$, $\mathcal{O}(0)^{ch}$ is precisely $\mathcal{D}_{\mathbb{P}^1}^{ch}$, and has been known since [MSV].

We prove that

(i) $H^0(\mathbb{P}^1, \mathcal{O}(n)^{ch})$ and $H^1(\mathbb{P}^1, \mathcal{O}(n)^{ch})$ are isomorphic to the irreducible $\widehat{sl}_2$-module at the critical level with highest weight n if $n \geq 0$;

(ii) $H^0(\mathbb{P}^1, \mathcal{O}(n)^{ch})$ and $H^1(\mathbb{P}^1, \mathcal{O}(n)^{ch})$ are isomorphic to the irreducible $\widehat{sl}_2$-module at the critical level with highest weight $-n-2$ if $n \leq -2$;

(iii) $H^0(\mathbb{P}^1, \mathcal{O}(n)^{ch}) = H^1(\mathbb{P}^1, \mathcal{O}(n)^{ch}) = 0$ *if* $n = -1$*.*

This result is a direct generalization of [MSV], Theorem 5.7, sect.5.8, and our construction verifies the proposals made in [MSV], sect.5.15, one of the starting points of the present work.

To conclude, one can say that the category of $\mathcal{D}^{ch,tw}_{G/B}$-modules appears to be a cross between the Bernstein-Beilinson [BB1, BB2] localization of $\mathfrak{g}$-modules to the flag manifold and localization of $\widehat{\mathfrak{g}}$-modules at the critical level to the semi-infinite flag manifold. We hope that this point of view may prove useful.

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2. Preliminaries.

We will recall the basic notions of vertex algebra and describe computational tools to be used in the sequel.

All vector spaces will be over $\mathbb{C}$. All spaces are even.

2.1. **Definitions.** Let V be a vector space.

A *field* on V is a formal series

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \in (\mathrm{End}V)[[z, z^{-1}]]$$

such that for any $v \in V$ one has $a_{(n)}v = 0$ for sufficiently large n.

Let $Fields(V)$ denote the space of all fields on V.

A *vertex algebra* is a vector space V with the following data:

- a linear map $Y : V \to Fields(V)$, $V \ni a \mapsto a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$
- a vector $|0\rangle \in V$, called *vacuum vector*
- a linear operator $\partial : V \to V$, called *translation operator*

that satisfy the following axioms:

(1) (Translation Covariance)
$(\partial a)(z) = \partial_z a(z)$

(2) (Vacuum)
$|0\rangle(z) = \mathrm{id}$;
$a(z)|0\rangle \in V[z]$ and $a_{(-1)}|0\rangle = a$

(3) (Borcherds identity)

$$\begin{aligned}(2.1)\qquad &\sum_{j\geq 0} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)} \\ &= \sum_{j\geq 0} (-1)^j \binom{n}{j} \{a_{(m+n-j)}b_{(k+j)} - (-1)^n b_{(n+k-j)}a_{(m+j)}\}\end{aligned}$$

A vertex algebra V is *graded* if $V = \oplus_{n\geq 0} V_n$ and for $a \in V_i$, $b \in V_j$ we have

$$a_{(k)} b \in V_{i+j-k-1}$$

for all $k \in \mathbb{Z}$. (We put $V_i = 0$ for $i < 0$.)

We say that a vector $v \in V_m$ has *conformal weight* m and write $\Delta_v = m$.

If $v \in V_m$ we denote $v_k = v_{(k-m+1)}$, this is the so-called conformal weight notation for operators. One has

$$v_k V_m \subset V_{m-k}.$$

A *morphism* of vertex algebras is a map $f : V \to W$ that preserves vacuum and satisfies $f(v_{(n)} v') = f(v)_{(n)} f(v')$.

A *module* over a vertex algebra V is a vector space M together with a map

$$Y^M : V \to Fields(M),\ a \to Y^M(a,z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1}, \tag{2.2}$$

that satisfy the following axioms:

(1) $|0\rangle^M(z) = \mathrm{id}\ _M$

(2) (Borcherds identity)

$$\begin{aligned} &\sum_{j\geq 0} \binom{m}{j} (a_{(n+j)} b)^M_{(m+k-j)} \\ &= \sum_{j\geq 0} (-1)^j \binom{n}{j} \{ a^M_{(m+n-j)} b^M_{(k+j)} - (-1)^n b^M_{(n+k-j)} a^M_{(m+j)} \} \end{aligned} \tag{2.3}$$

Note that we have unburdened the notation by letting

$$a^M(z) = Y^M(a,z).$$

A module M over a graded vertex algebra V is called *graded* if $M = \oplus_{n\geq 0} M_n$ with $v_k M_l \subset M_{l-k}$ (assuming $M_n = 0$ for negative n).

A *morphism of modules* over a vertex algebra V is a map $f : M \to N$ that satisfies $f(v^M_{(n)} m) = v^N_{(n)} f(m)$ for $v \in V$, $m \in M$. f is *homogeneous* if $f(M_k) \subset N_k$ for all k.

2.2. **Examples.**

2.2.1. *Affine vertex algebras.* Let $\mathfrak{g}$ be a semisimple Lie algebra and $\langle , \rangle : S^2\mathfrak{g} \to \mathbb{C}$ an invariant form on $\mathfrak{g}$. The *affine Lie algebra* $\hat{\mathfrak{g}}$ associated with $\mathfrak{g}$ and $\langle , \rangle$ is a central extension of $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$ defined as follows. As a vector space, $\hat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$ and the Lie bracket is

$$[x\otimes t^n, y\otimes t^m] = [x,y]\otimes t^{m+n} + n\delta_{n+m,0}\langle x,y\rangle K$$

K is a central element.

We denote $x\otimes t^n$ by x_n and write $x(z) = \sum x_n z^{-n-1}$.

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be a Cartan decomposition of $\mathfrak{g}$.

Denote $\hat{\mathfrak{g}}_< = \mathfrak{g}\otimes t\mathbb{C}[t]$, $\hat{\mathfrak{g}}_> = \mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}]$ and $\hat{\mathfrak{g}}_\leq = \mathfrak{g}\otimes\mathbb{C}[t]\oplus\mathbb{C}K$.

Define $\hat{\mathfrak{g}}_+ = \mathfrak{n}_+ \oplus \hat{\mathfrak{g}}_>$, $\hat{\mathfrak{g}}_- = \mathfrak{n}_- \oplus \hat{\mathfrak{g}}_<$. Then $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \mathfrak{h} \oplus \mathbb{C}K \oplus \hat{\mathfrak{g}}_-$.

The space of invariant forms is one-dimensional, and we will let $\langle , \rangle$ be that form for which $(\theta,\theta) = 2$ where θ is the longest root.

Introduce the following induced module

$$V_k(\mathfrak{g}) = \mathrm{Ind}\ ^{\hat{\mathfrak{g}}}_{\hat{\mathfrak{g}}_\leq} \mathbb{C}_k, \tag{2.4}$$

where $\mathbb{C}_k$ is a 1-dimensional $\hat{\mathfrak{g}}_{\le}$-module generated by a vector v_k such that $\hat{\mathfrak{g}}_{<}v_k = 0$, $\mathfrak{g}v_k = 0$ and $Kv_k = v_k$.

$V_k(\mathfrak{g})$ carries a vertex algebra structure that is defined by assigning to $x_{-1}v_k$ $x \in \mathfrak{g}$, the field $x(z) = \sum x_n z^{-n-1}$. These fields generate $V_k(\mathfrak{g})$.

$V_k(\mathfrak{g})$ is a graded vertex algebra with generators having conformal weight 1. For example,

$$V_k(\mathfrak{g})_0 = \mathbb{C}_k,\ V_k(\mathfrak{g})_1 = \mathfrak{g} \otimes t^{-1}v_k. \tag{2.5}$$

2.2.2. *Commutative vertex algebras.* A vertex algebra is said to be *commutative* if $a_{(n)}b = 0$ for a, b in V and $n \ge 0$. The structure of a commutative vertex algebras is equivalent to one of commutative associative algebra with a derivation.

If W is a vector space we denote by H_W the algebra of differential polynomials on W. As an associative algebra it is a polynomial algebra in variables x_i, ∂x_i, $\partial^{(2)}x_i$, $\ldots$ where $\{x_i\}$ is a basis of W^*. A commutative vertex algebra structure on H_W is uniquely determined by attaching the field $x(z) = e^{z\partial}x_i$ to $x \in W^*$.

H_W is equipped with grading such that

$$(H_W)_0 = \mathbb{C},\ (H_W)_1 = W^*. \tag{2.6}$$

2.2.3. *Beta–gamma system.* Define the Heisenberg Lie algebra to be the algebra with generators a^i_n, b^i_n, $1 \le i \le N$ and K that satisfy $[a^i_m, b^j_n] = \delta_{m,-n}\delta_{i,j}K$, $[a^i_n, a^j_m] = 0$, $[b^i_n, b^j_m] = 0$.

Its Fock representation M is defined to be the module induced from the one-dimensional representation $\mathbb{C}_1$ of its subalgebra spanned by a^i_n, $n \ge 0$, b^i_m, $m > 0$ and K with K acting as identity and all the other generators acting as zero.

The *beta-gamma system* has M as an underlying vector space, the vertex algebra structure being determined by assigning the fields

$$a^i(z) = \sum a^i_n z^{-n-1}, \quad b^i(z) = \sum b^i_n z^{-n}$$

to $a^i_{-1}1$ and $b^i_0 1$ resp., where $1 \in \mathbb{C}_1$.

This vertex algebra is given a grading so that the degree of operators a^i_n and b^i_n is n. In particular,

$$M_0 = \mathbb{C}[b^1_0, ..., b^N_0],\ M_1 = \bigoplus_{j=1}^{N}(b^j_{-1}M_0 \oplus a^j_{-1}M_0). \tag{2.7}$$

2.3. **Vertex algebroids.**

2.3.1. *Definition.* Let V be a graded vertex algebra. We briefly recall from [GMS1] basic results on the structure that is induced by vertex operations on the subspace $V_{\le 1} = V_0 + V_1$.

Let us define a 1-truncated vertex algebra to be a sextuple $(V_0 \oplus V_1, |0\rangle, \partial, {}_{(-1)}, {}_{(0)}, {}_{(1)})$ where the operations ${}_{(-1)}, {}_{(0)}, {}_{(1)}$ satisfy all the axioms of a vertex algebra that make sense upon restricting to the subspace $V_0 + V_1$. (The precise definition can be found in [GMS1]). The category of 1-truncated vertex algebras will be denoted $\mathcal{V}ert_{\le 1}$.

The notion of a 1-truncated vertex algebra is equivalent to that of a vertex algebroid. For the definition the reader is referred to [GMS1]; in this note we only recall the main ingredients and properties of a vertex algebroid.

For a graded vertex algebra V, set $A = V_0$, $\Omega = A_{(-1)}\partial A$ and $T = V_1/\Omega$. The axioms of vertex algebra imply the following:

(1) $A = V_0$ is a commutative associative algebra with respect to ${}_{(-1)}$;
(2) Ω is an A-module via $a \cdot \omega = a_{(-1)}\omega$ and the translation map $\partial : A \to \Omega$ is a derivation;
(3) $T = V_1/\Omega$ is a Lie algebra with bracket ${}_{(0)}$ and a left A-module via ${}_{(-1)}$;
(4) Ω is a T-module with the action induced by ${}_{(0)}$;
(5) the map ${}_{(0)} : T \times A \to A$ defines an action of T on A by derivations.
(6) the maps ${}_{(1)} : T \times \Omega \to A$ and ${}_{(1)} : \Omega \times T \to A$ are A-bilinear pairings that satisfy $\tau_{(1)}\omega = \omega_{(1)}\tau$ and are determined by $\tau_{(1)}\partial a = \tau_{(0)}a$.

The gadget (1)–(6) is quite classical; in particular, (1,5) mean that T is an A-*Lie algebroid* [BB2]. Altogether (1–6) were called an *extended Lie algebroid* in [GMS1], a concept that is equivalent to that of a *Courant algebroid* — this is a remark of P. Bressler, [Bre].

All of the vertex algebra structure on $V_0 + V_1$ comprises more data than (1–6), but not much more. A *vertex algebroid* is a quintuple $(A \oplus T \oplus \Omega, \partial, \gamma, \langle , \rangle, c)$ where $(A, \Omega, T, \partial$ are as in (1–5), $\gamma : A \times T \to \Omega$ is a bilinear map,

$$\langle , \rangle : (T \oplus \Omega) \times (T \oplus \Omega) \to A$$

is a symmetric bilinear pairing, and

$$c : T \times T \to \Omega$$

is a skew-symmetric bilinear pairing. These data satisfy a list of axioms to be found in [GMS1]. We will not record those axioms here – they are a result of writing down the restriction of the Borcherds identity to conformal weights 0 and 1 subspaces – but we will supply the reader with a short dictionary:

Fix a splitting $V_1 = T \oplus \Omega$, see item (3) above.

The map γ is determined by the classical data (1–6), the splitting chosen, and the Borcherds identity.

The pairing $\langle , \rangle$ is an extension of the pairing from item (6); the extra part is

$$\langle \xi, \eta \rangle = \xi_{(1)}\eta, \ \xi, \eta \in T. \tag{2.8}$$

The map c, the key to extending the classical data (1–6) to a vertex algebroid is defined by (in the presence of the splitting)

$$\xi_{(0)}\eta = [\xi, \eta] + c(\xi, \eta), \ \xi, \eta \in T. \tag{2.9}$$

The category of vertex algebroids, to be denoted $\mathcal{A}lg$, is defined in an obvious manner and is immediately seen to be equivalent to that of 1-truncated vertex algebras.

2.3.2. *Truncation and vertex enveloping algebra functors.* There is an obvious truncation functor

$$t : \mathcal{V}ert \to \mathcal{V}ert_{\leq 1}$$

that assigns to every vertex algebra a 1-truncated vertex algebra. This functor admits a left adjoint [GMS1]

$$u : \mathcal{V}ert_{\leq 1} \to \mathcal{V}ert$$

called a *vertex enveloping algebra functor*.

In the context of vertex algebroids, these functors become $\mathcal{A} : \mathcal{V}ert \to \mathcal{A}lg$ and its left adjoint $U : \mathcal{A}lg \to \mathcal{V}ert$.

2.3.3. *Examples.* Various examples of vertex algebras reviewed above are, in fact, vertex enveloping algebras of appropriate vertex algebroids:

- in the situation of sect. 2.2.1, $\mathbb{C}_k \oplus \mathfrak{g} \otimes t^{-1} v_k$ is a vertex algebroid, see (2.5), and $V_k(\mathfrak{g}) = U(\mathbb{C}_k \oplus \mathfrak{g} \otimes t^{-1} v_k, ...)$;
- in the situation of sect. 2.2.2, $\mathbb{C} \oplus W^*$ is a vertex algebroid – obviously commutative, see (2.6), and $H_W = U(\mathbb{C} \oplus W^*)$;
- in the situation of sect.2.2.3, $M_0 \oplus M_1$ is a vertex algebroid, see (2.7), and $M = U(M_0 \oplus M_1)$.

If we have two vertex algebras, V, W, then their tensor product $V \otimes W$ carries a vertex algebra structure defined as usual, see e.g. [K, FBZ], by letting

$$(v \otimes w)_{(n)} a \otimes b = \sum_{j=-\infty}^{+\infty} v_{(j)} a \otimes w_{(n-j-1)} b. \tag{2.10}$$

One similarly defines the tensor product of two vertex algebroids. In the more convenient language of 1-truncated vertex algebras, if $V = V_0 \oplus V_1$, $W = W_0 \oplus W_1$ are two 1-truncated vertex algebras, then we define

$$V \overset{\bullet}{\otimes} W = (V_0 \otimes W_0) \oplus (V_0 \otimes W_1 \oplus V_1 \otimes W_0), \tag{2.11}$$

$$(v \otimes w)_{(n)} a \otimes b = \sum_j v_{(j)} a \otimes w_{(n-j-1)} b, \tag{2.12}$$

where unlike (2.10) the summation $\sum_j$ is extended to those j for which it makes sense.

The vertex algebroid $M_0 \oplus M_1$ will give rise to the simplest example of an algebra of chiral differential operators, the subject to which we now turn, and the tensor product $(M_0 \oplus M_1) \overset{\bullet}{\otimes} (\mathbb{C} \oplus W^*)$ will be similarly used to construct an algebra of twisted chiral differential operators in sect. 4.3.1; here $\mathbb{C} \oplus W^*$ is a commutative vertex algebroid from sect. 2.3.3.

2.4. **Chiral differential operators.** A vertex algebra V is called an *algebra of chiral differential operators* over A, CDO for short, if V is the vertex envelope of a vertex algebroid $\mathcal{A} = A \oplus T \oplus \Omega$ such that $T = \mathrm{Der}\ A$ and $\Omega = \Omega^1_A$, the module of Kähler differentials.

An algebra of chiral differential operators over A does not exist for any A, but it does exist locally on $SpecA$.

To be more precise, a smooth affine variety $U = \mathrm{Spec}A$ will be called *suitable for chiralization* if $Der(A)$ is a free A-module admitting an abelian frame $\{\tau_1, ..., \tau_n\}$. In this case there is a CDO over A, which is uniquely determined by the condition that $(\tau_i)_{(1)}(\tau_j) = (\tau_i)_{(0)}(\tau_j) = 0$; in other words we let the "quantum data", $\langle , \rangle$ and c vanish on the basis vector fields, cf. (2.8,2.9). Denote this CDO by $D^{ch}_{U,\tau}$.

Theorem 2.1. *Let $U = \mathrm{Spec}A$ be suitable for chiralization with a fixed abelian frame $\{\tau_i\} \subset DerA$.*

(i) For each closed 3-form $\alpha \in \Omega^{3,cl}_A$ there is a CDO over A that is uniquely determined by the conditions

$$(\tau_i)_{(1)} \tau_j = 0,\ (\tau_i)_{(0)} \tau_j = \iota_{\tau_i} \iota_{\tau_j} \alpha.$$

Denote this CDO by $\mathcal{D}_{U,\tau}(\alpha)$.

(ii) Each CDO over A is isomorphic to $\mathcal{D}_{U,\tau}(\alpha)$ for some α.

(iii) $\mathcal{D}_{U,\tau}(\alpha_1)$ and $\mathcal{D}_{U,\tau}(\alpha_2)$ are isomorphic if and only if there is $\beta \in \Omega^2_A$ such that $d\beta = \alpha_1 - \alpha_2$. In this case the isomorphism is determined by the assignment $\tau_i \mapsto \tau_i + \iota_{\tau_i}\beta$.

If $A = \mathbb{C}[x_1, ..., x_n]$, one can choose $\partial/\partial x_j$, $j = 1, ..., n$, for an abelian frame and check that the beta-gamma system M of sect. 2.2.3 is a unique up to isomorphism CDO over $\mathbb{C}^n$. A passage from M to Theorem 2.1 is accomplished by the identifications $b^j_0 1 = x_j$, $a^j_{-1}1 = \partial/\partial x_j$.

The construction of CDOs from Theorem 2.1 can be sheafified; however, what one gets is not a sheaf but rather a gerbe over a smooth variety X bound by the complex $\Omega^2_X \to \Omega^{3,cl}_X$. The existence of global objects in this gerbe depends on vanishing of a certain characteristic class of X. For the precise description of the situation we refer the reader to [GMS1], here let us just note that for a smooth variety X with vanishing first Pontrjagin class there exist such sheaves; they are called *sheaves of chiral differential operators.*

Let $\mathcal{D}^{ch}_X$ denote any of such sheaves. This is a graded sheaf. A straightforward consequence of the construction is that $(\mathcal{D}^{ch}_X)_0 \simeq \mathcal{O}_X$, $\Omega^1_X \subset (\mathcal{D}^{ch}_X)_1$ and $(\mathcal{D}^{ch}_X)_1/\Omega^1_X \simeq \mathcal{T}_X$.

Another "classical" object that we attach to a vertex algebra V is the universal enveloping algebra U_AT of the A-Lie algebroid T. By definition, U_AT is the quotient of the tensor algebra

$$Tens(A \oplus T) = \oplus_{i \geq 0}(A \oplus T)^{\otimes i}$$

modulo the ideal R generated by the elements $a \otimes b - ab$, $\tau \otimes a - a \otimes \tau - \tau(a)$, $\tau \otimes \xi - \xi \otimes \tau - [\tau, \xi]$, $a \otimes \tau - a\tau$, $1_A - 1_{\mathbb{C}}$. In the next section we will see how this algebra appears via Zhu's construction.

3. Zhu's correspondence

The work [Zhu] revealed a beautiful and nontrivial connection between the world of vertex algebras and that of associative algebras. The main result of Zhu's theory states that to each graded vertex algebra V one can naturally attach an associative algebra, to be denoted $Zhu(V)$, such that there is a one-to-one correspondence between simple $Zhu(V)$-modules and simple V-modules.

The aim of this section is to prove the theorem below. This is a statement connecting Zhu algebra of V to the universal enveloping algebra U_AT defined in the previous section. The basic observation is that there is a natural associative algebra morphism

$$\alpha : U_AT \to Zhu(V), \tag{3.1}$$

to be constructed in Subsection 3.2.1.

Theorem 3.1. *(1) If V is generated by $V_0 + V_1$, then the map α is surjective.*

(2) If V is a vertex enveloping algebra of $V_0 + V_1$ and T and Ω are free A-modules, then α is an isomorphism.

Remark 3.2. If $V = V_k(\mathfrak{g})$, see sect. 2.2.1, then $U_AT = U\mathfrak{g}$, and Theorem 3.1 follows from the isomorphism

$$U\mathfrak{g} \simeq Zhu(V_k(\mathfrak{g}))$$

established in [FZ]. It is fair to say that Theorem 3.1 is a variation on the theme of [FZ].

As a corollary, we will have a description of the sheaf of Zhu algebras for the vertex algebra of twisted differential operators.

The proofs of many of the auxiliary results below can be found elsewhere, e.g. [Zhu, FZ, R, MZ, L].

3.1. **Definition of the Zhu algebra.**

3.1.1. *Motivation.* While vertex operations ${}_{(n)}$, $n \in \mathbb{Z}$ satisfy axioms as remote from associativity as Borcherds identity, the endomorphisms $v_{(n)}$ belong to an associative algebra $\mathrm{E}nd\, V$. Furthermore, if M a graded V-module, then there are maps

$$V \to \mathrm{E}nd\,(M),\ v \mapsto v_0^M$$

and by restriction

$$V \to \mathrm{E}nd\,(M_0),\ v \mapsto v_0^M|_{M_0}$$

Now one can ask if there is an operation $*$ on V that makes the latter an algebra morphism for any M. The answer is yes, and in order to find such an operation let us look at the Borcherds identity for a V-module M:

$$\sum_{j\geq 0}\binom{m}{j}(a_{(n+j)}b)^M_{(m+k-j)} = \sum_{j\geq 0}(-1)^j\binom{n}{j}\{a^M_{(m+n-j)}b^M_{(k+j)} - (-1)^n b^M_{(n+k-j)}a^M_{(m+j)}\}$$

in the conformal weight notation

$$\sum_{j\geq 0}\binom{m}{j}(a_{(n+j)}b)^M_{m+k-\Delta_a-\Delta_b+n+2} = \sum_{j\geq 0}(-1)^j\binom{n}{j}\{a^M_{m+n-j-\Delta_a+1}b^M_{k+j-\Delta_b+1}$$

$$-(-1)^n b^M_{n+k-j-\Delta_b+1}a^M_{m+j-\Delta_a+1}\}$$

and consider the case when $m = \Delta_a$, $n = -1$, and both sides are degree 0 morphisms, which requires $k = \Delta_b - 1$. We obtain

$$\sum_{j\geq 0}\binom{\Delta_a}{j}(a_{(-1+j)}b)^M_0 = \sum_{j\geq 0}(a^M_{-j}b^M_j + b^M_{-j-1}a^M_{j+1})$$

Restricting this to M_0 will give us

$$(\sum_{j\geq 0}\binom{\Delta_a}{j}a_{(-1+j)}b)^M_0|_{M_0} = a^M_0 b^M_0|_{M_0}$$

which means that for the desired operation we can take the following

$$a * b = \sum_{j=0}^{\Delta_a}\binom{\Delta_a}{j}a_{(-1+j)}b.$$

This operation is not associative. However, it is shown in Zhu's work that there is a subspace $O(V) \subset V$ that is an ideal with respect to this operation and acts as zero on M_0 for each M, and such that the induced multiplication on $V/O(V)$ is associative. Specifically, $O(V) = (\partial + H)V * V$, where $Hv = \Delta_v v$ for homogeneous v. It is straightforward to verify that $v_0 = 0$ for $v \in O(V)$, as $(\partial v)_0 = -\Delta_v v_0$. What is more remarkable is that $O(V)$ is an ideal with respect to $*$ and that $*$ is associative modulo $O(V)$. Furthermore, the associative algebra $(V/O(V), *)$ carries some essential information on the category of V-modules.

3.1.2. *Formal definition.*

Definition 3.3. For homogeneous $a \in V$ define the Zhu multiplication

$$a * b = \sum_{i=0}^{\Delta_a} \binom{\Delta_a}{i} a_{(i-1)} b.$$

More generally, for $n \in \mathbb{Z}$ define

$$a *_n b = \sum_{i=0}^{\Delta_a} \binom{\Delta_a}{i} a_{(n+i)} b$$

To make this operation associative one has to pass over to a properly chosen quotient of $(V, *)$.

Denote $O(V) = V *_{-2} V$. One can show that $O(V) = V * dV = dV * V$ where $d = \partial + H$

One has the following

Proposition 3.4. *(1) $V * O(V) \subset O(V)$, $O(V) * V \subset O(V)$*
(2) $a * (b * c) - (a * b) * c \in O(V)$ for all a, b, c in V.

Definition 3.5. Define the Zhu algebra to be the space

$$Zhu(V) = V/O(V)$$

endowed with the multiplication induced by $*$.

It follows from the proposition above that the Zhu algebra is an associative algebra. It is naturally a filtered algebra with the filtration induced by the conformal weight filtration of the vertex algebra V. Specifically, $F^m Zhu(V) = \pi(\bigoplus_{i=0}^{m} V_i)$ where $\pi : V \to V/O(V)$ is the natural projection map.

Let $\mathcal{V}ert$ denote the category whose objects are $\mathbb{Z}_{\le 0}$-graded vertex algebras and the morphisms are graded vertex algebra maps.

The correspondence $V \mapsto Zhu(V)$ provides a functor from $\mathcal{V}ert$ to the category of filtered associative algebras.

3.2. **Relation of $Zhu(V)$ to $U_A T$.**

3.2.1. Recall that $A = V_0$ is an associative commutative algebra with multiplication $_{(-1)}$ and $T = V_1/V_{0(-1)}\partial V_0$ is an A-Lie algebroid.

Since $\Omega = A_{(-1)}\partial A = A * dA$ is a subset of $V * dV = O(V)$, we have a natural map $\bar{\alpha} : T = V_1/\Omega \to V/O(V) = Zhu(V)$. We denote $\bar{\alpha}(\tau) = \bar{\tau}$.

Lemma 3.6. *A is naturally embedded into $Zhu(V)$.*

Proof. First, notice that elements of the form $dw * v$ do not have a degree 0 component. Indeed, the lowest conformal weight summand in $dw * v \in \oplus_{i=\Delta_v}^{\Delta_v+\Delta_w+1} V_i$ is equal to $(dw)_0 v = 0$ (since $d = \partial + H$ and $(\partial w)_0 = -\Delta_w w_0$).

Thus $V_0 \cap O(V) = 0$ and the restriction of the projection $\pi : V \to Zhu(V)$ to V_0 is injective. Since $a * b = a_{(-1)} b$ for $a, b \in V_0$, this is an algebra embedding. □

Lemma 3.7. *The natural map $\bar{\alpha} : T \to Zhu(V)$ extends to an algebra homomorphism $\alpha : U_A T \to Zhu(V)$*

Proof. The inclusion $A \hookrightarrow Zhu(V)$ and the map $\bar{\alpha} : T \to Zhu(V)$ uniquely determine algebra morphism $Tens(A \oplus T) \to Zhu(V)$.

We have an exact sequence

$$0 \to R \to Tens(A \oplus T) \to U_A T \to 0,$$

cf. the end of sect. 2.4

Under the map $Tens(A \oplus T) \to Zhu(V)$ the generators of R are mapped to $R_1 = a*b - a_{(-1)}b$, $R_2 = \bar{\tau}*a - a*\bar{\tau} - \tau(a)$, $R_3 = \bar{\tau}*\bar{\xi} - \bar{\xi}*\bar{\tau} - \overline{[\tau,\xi]}$, $R_4 = a*\bar{\tau} - a_{(-1)}\bar{\tau}$, and $1-1$. To finish the proof, it suffices to show that $R_i = 0$, $i = 1, 2, 3, 4$.

1) $R_1 = 0$ due to the algebra inclusion $A \hookrightarrow Zhu(V)$.

2). Let t denote any lifting of τ to V_1. We check that $t*a - a*t - \tau(a) \in O(V)$

Recall that for $x \in V_1$ the Zhu operation reduces to $x*v = x_{(-1)}v + x_{(0)}v$ and by definition, $\tau(a) = t_{(0)}a$. Hence

$$t*a - a*t - t(a) = t_{(-1)}a + t_{(0)}a - a_{(-1)}t - t_{(0)}a =$$

$$= t_{(-1)}a - a_{(-1)}t = [t_{(-1)}, a_{(-1)}]|0\rangle = \partial(t_{(0)}a) \in \partial(V_0) \subset O(V)$$

3) Let q be any lifting of $[\tau,\xi]$ to V_1. Then $q - t_{(0)}x \in \Omega$ for any lifting x of ξ and t of τ. Thus it suffices to check

$$t*x - x*t - t_{(0)}x = t_{(-1)}x + t_{(0)}x - x_{(-1)}t - x_{(0)}t - t_{(0)}x =$$

$$= [t_{(-1)}, x_{(-1)}]|0\rangle - x_{(0)}t = \partial(t_{(0)}x) - x_{(0)}t = \partial(t_{(0)}x) + t_{(0)}x - \partial(t_{(1)}x) \in O(V)$$

4) $R_4 = 0$ follows from $a*v = a_{(-1)}v$ for any $v \in V$, $a \in A$.

Thus, the map factors through $U_A T$. $\square$

3.3. **Properties of the Zhu algebra when V is generated by $V_{\leq 1}$.**

3.3.1. Notice that

$$a *_n v = a_{(n)}v, \quad \text{for } a \in V_0 \tag{3.2}$$

$$x *_n v = x_{(n)}v + x_{(n+1)}v \quad \text{for } x \in V_1 \tag{3.3}$$

We will start with deriving a different presentation of the ideal $O(V)$. First, we observe the following:

Proposition 3.8. *(1) $V *_n V \subset V *_{n+1} V$ for all $n \neq -1$*

Proof. Straightforward, see e.g. [R] $\square$

Consequently, $V *_n V \subset O(V)$ for all $n \leq -2$. In particular, the subspace

$$O'(V) = \text{span}\{a_{(n)}v, (x_{(n)} + x_{(n+1)})v|\ n \leq -2,\ a \in V_0, x \in V_1, v \in V\} \tag{3.4}$$

lies in $O(V)$.

The aim of the next lemmas is to show that if V is generated by $V_0 + V_1$ then $O'(V)$ is in fact all of $O(V)$.

First, we notice that $O'(V)$ is invariant under the action of f_m for $f \in V_{\leq 1}$, $m \leq 0$.

Lemma 3.9. *We have*

$$b_{(m)}O'(V) \subset O'(V), \quad y_{(n)}O'(V) \subset O'(V)$$

for any $b \in V_0$, $y \in V_1$, $m \leq -1$, $n \leq 0$.

Proof. The space $O'(V)$ is spanned by elements of the form $a_{(k)}v$ and $(x_{(k)}+x_{(k+1)})v$, $k\leq -2$. We need to show that for $a,b\in V_0$ and $x,y\in V_1$ the elements $A=b_{(m)}a_{(k)}v,\ B=b_{(m)}(x_{(k)}+x_{(k+1)})v,\ C=y_{(n)}a_{(k)}v,\ D=y_{(n)}(x_{(k)}+x_{(k+1)})v$ can also be written in such form.

Cases A and B. For $m\leq -2$ we have $b_{(m)}O'(V)\subset b_{(m)}V\subset O'(V)$ by definition of $O'(V)$. For $m=-1$ we use the commutator identities. In one case we have

$$A=b_{(-1)}a_{(k)}v=a_{(k)}b_{(-1)}v+[b_{(-1)},a_{(k)}]v=a_{(k)}b_{(-1)}v\in O'(V),$$

in the other we use $[b_{(-1)},x_{(k+1)}]v=(b_{(0)}x)_{(k)}v\in O'(V),\ k\leq -2$ to conclude that

$$B=b_{(-1)}(x_{(k)}+x_{(k+1)})v=(x_{(k)}+x_{(k+1)})b_{(-1)}v\ +(b_{(0)}x)_{(k-1)}v+(b_{(0)}x)_{(k)}v$$

is also in $O'(V)$.

The remaining cases are also implied by the commutator formula:

$$C=y_{(n)}a_{(k)}v=a_{(k)}y_{(n)}v+[y_{(n)},a_{(k)}]v=a_{(k)}y_{(n)}v+(y_{(0)}a)_{(n+k)}v\in O'(V)$$

for $k\leq -2$, $n\leq 0$.

$$D=y_{(n)}(x_{(k)}+x_{(k+1)})v=(x_{(k)}+x_{(k+1)})y_{(n)}v+[y_{(n)},x_{(k)}+x_{(k+1)}]v$$

To show $D\in O'(V)$ we notice

$$\begin{aligned}[y_{(n)},x_{(k)}+x_{(k+1)}]v&=((y_{(0)}x)_{(n+k)}+(y_{(0)}x)_{(n+k+1)})v+\\&+n(y_{(1)}x)_{(n+k-1)}v+n(y_{(1)}x)_{(n+k)}v=\\&=(y_{(0)}x)*_{n+k}v+n(y_{(1)}x)*_{n+k-1}v+n(y_{(1)}x)*_{n+k}v,\end{aligned}$$

all three terms in $O'(V)$. $\square$

From now on, V is a vertex algebra generated by $V_{\leq 1}$, unless stated otherwise.

Lemma 3.10. *$O(V)=O'(V)$.*

Proof We need to show that $u*_{-2}v\in O'(V)$ for all $u,v\in V$. Following the proof of Rosellen ([R], Proposition 6.2.5) we show that $u*_n v\in O'(V)$ for all $n\leq -2$ by induction on Δ_u.

Basis of induction: $\Delta_u=0$, that is, $u\in A$. Then $u*_n v\in O'(V)$ for all $n\leq -2$ by definition of $O'(V)$.

Suppose we showed that $u*_n v\in O'(V)$ for all $u,v\in V$ such that $\Delta_u=k$ and all $n\leq -2$. Now we need to prove it for V_{k+1}. Any element of V_{k+1} is a sum of elements of the form $x_{(-r)}u$ or $b_{(-r-1)}u$ where $\Delta_u\leq k$, $r\geq 1$.

Consider $b_{(-r-1)}u$ with $b\in V_0$. Since $b_{(-r-1)}=b*_{-r-1}$, we can use the associativity formula (see [R, Proposition 6.2.2])

$$\begin{aligned}(b*_{-r-1}u)*_{-2-t}v=\sum_{i,j=0}^{\infty}(-1)^i\binom{r}{j}\binom{-r-1}{i}&(b*_{-r-1-i}(u*_{-2-t+i+j}v)\\&-(-1)^{-r-1}u*_{-3-t-r-i+j}(b*_i v)))\end{aligned}$$

The term $b*_{-r-1-i}(u*_{-2-t+i+j}v)$ is in $O'(V)$ since $-r-1-i\leq -2$.

For the second term, we can assume that $0\leq j\leq r$ since otherwise $\binom{r}{j}=0$ (as $r\geq 1$). Then $-3-t-r-i+j\leq -3-t-i\leq -3$ and the term is in $O'(V)$ by the induction assumption.

The proof for the case $x_{(m)}u$ repeats Rosellen's proof in Proposition 6.2.5. For the sake of completeness, we reproduce it here, notation slightly changed.

We need to show that $(x_{(m)}u) *_n v \in O'(V)$ for all $m \leq -1$, $n \leq -2$. By (3.3) we have

$$(x_m u) *_n v = (x *_m u) *_n v - (x_{m+1}u) *_n v. \tag{3.5}$$

The last term of r.h.s. is in $O'(V)$ by the induction assumption.

By [R], Proposition 6.2.2 we have

$$(x *_m u) *_n v = \sum_{i,j \geq 0} (-1)^i \binom{-m-1}{j} \binom{m}{i} (x *_{m-i} (u *_{n+i+j} v) - (-1)^m u *_{n+j+m-i} (x *_i v))$$

By induction, $u *_{n+j+m-i} (x *_i v) \in O'(V)$ since $j \leq -m-1$.

From (3.3) it follows that $x *_{m-i} (u *_{n+i+j} v) \in O'(V)$ if $m - i \leq -2$, that is, if $m \leq -2$ or $i > 0$. If $m = -1$ and $i = 0$ then $j = 0$. By the induction assumption, $u *_n v \in O'(V)$. Thus, using Lemma 3.9 we have $x *_{-1} (u *_n v) = x_{(-1)}(u *_n v) + x_{(0)}(u *_n v) \in O'(V)$. The lemma is proved. $\square$

Corollary 3.11. $O(V) = \text{span}\{\omega_{(n+1)}v,\ (x_{(n)} + x_{(n+1)})v,\ v \in V,\ n \leq -2,\ x \in V_1,\ \omega \in \Omega\}$

Proof. Denote

$$O''(V) = \text{span}\{\omega_{(n+1)}v,\ (x_{(n)} + x_{(n+1)})v,\ v \in V,\ n \leq -2\}.$$

We show that $O'(V) = O''(V)$.

Clearly, $O'(V) \subset O''(V)$, since $a_{(n)} = -\frac{1}{n+1}(\partial a)_{(n+1)}$, $n \leq -2$.

Now check $O''(V) \subset O'(V)$. Let $\omega = a\partial b$, $a, b \in A$ and let $n \leq -1$. Then

$$\omega_{(n)}v = \sum_{j \geq 0} (\partial b)_{(n-1-j)} a_{(j)} v + a_{(-1)}(\partial b)_{(n)} v + \sum_{j \geq 1} a_{(-1-j)} (\partial b)_{(n+j)} v =$$

$$= \sum_{j \geq 0} (j + 1 - n) b_{(n-2-j)} a_{(j)} v - n a_{(-1)} b_{(n-1)} v + \sum_{j \geq 0} a_{(-2-j)} (\partial b)_{(n+1+j)} v =$$

Both sums clearly are in $O'(V)$. The middle term is in $O'(V)$ since $b_{(n-1)}v \in O'(V)$ and $a_{(-1)}O'(V) \subset O'(V)$, see Lemma 3.9. $\square$

3.3.2. Fix a vertex algebra V which is generated by $V_{\leq 1}$.

Using results obtained above we can state the following two lemmas.

Lemma 3.12. *$Zhu(V)$ is spanned by 1 and*

$$\pi(a_{(-1)} x^1_{(-1)} \dots x^k_{(-1)} |0\rangle), \text{ where } a \in V_0,\ x^i \in V_1,\ 1 \leq i \leq k,\ k \geq 0$$

where $\pi : V \to V/O(V)$ denotes the projection map.

Proof. Indeed, if V is generated by $V_{\leq 1}$, then V is spanned by V_0 and monomials of the form $a_{(m)} x^1_{(-p_1-1)} \dots x^k_{(-p_k-1)}$, $a \in V_0$, $x^i \in V_1$. All such monomials with $m \leq -2$ are by definition in $O'(V)$, so they have no contribution to $Zhu(V)$.

Since $(x_{(n-1)} + x_{(n)})v \in O'(V)$, $n \leq -1$ and $x_{(n)}O'(V) \subset O'(V)$ we can conclude that

$$a_{(-1)} x^1_{(-p_1-1)} \dots x^k_{(-p_k-1)} |0\rangle \equiv (-1)^{p_1} a_{(-1)} x^1_{(-1)} x^2_{(-p_2-1)} \dots x^k_{(-p_k-1)} |0\rangle \equiv$$

$$\equiv (-1)^{p_1+p_2} a_{(-1)} x^1_{(-1)} x^2_{(-1)} \dots x^k_{(-p_k-1)} |0\rangle \equiv \dots \equiv (-1)^{p_1+\dots p_k} a_{(-1)} x^1_{(-1)} \dots x^k_{(-1)}$$

modulo $O'(V)$ $\square$

Lemma 3.13. *$Zhu(V)$ is generated by the image of $V_{\leq 1}$.*

Proof. In view of the previous lemma, we just need to show that the elements $\pi(a_{(-1)}x^1_{(-1)}\dots x^k_{(-1)}|0\rangle)$ are products of elements of $\pi(V_{\leq 1})$. We have $a*v = a_{(-1)}v$ from definition and $v*x \equiv x_{(-1)}v \mod O(V)$ from [Zhu], Lemma 2.1.3. Therefore

$$x^r * \cdots * x^2 * x^1 \equiv {x^1}_{(-1)}(x^r * \cdots * x^2) \equiv \cdots \equiv {x^1}_{(-1)}{x^2}_{(-1)}\dots x^r$$

and thus $a * x^r * \cdots * x^2 * x^1 \equiv a_{(-1)}{x^1}_{(-1)}{x^2}_{(-1)}\dots x^r \mod O(V)$. □

3.3.3. *Filtration of the Zhu algebra.* Here we briefly recall different filtrations of V that were dealt with in [GMS1] and the corresponding induced filtrations on $Zhu(V)$.

Let V be a $\mathbb{Z}_{\geq 0}$-graded vertex algebra generated by its first two components. There is an obvious *conformal weight filtration* $\mathcal{F} = \{F^nV, n \geq 0\}$ defined by

$$F^nV = \bigoplus_{i=0}^{n} V_i$$

In addition, there is a natural filtration $\mathcal{G} = \{G^nV,\ n \geq 0\}$ by "number of vector fields". By a vector field we mean an element of $T = V_1/\Omega$; by abuse of language we will call a vector field any element of V_1 that projects onto a nontrivial element of $T = V_1/\Omega$.

This filtration is defined as follows: the space G^mV is the space spanned by monomials $s^1_{(n_1)}\dots s^r_{(n_r)}|0\rangle$, where s^i are elements either of V_0 or of V_1, $n_i \leq -1$, and the number of vector fields among $s^1, \dots, s^r$ is less than or equal to m, i.e. $|\{i:\ s_i \in V_1 \backslash \Omega\}| \leq m$.

Clearly, $\mathcal{G}$ is an increasing exhaustive filtration of V.

Lemma 3.14. *Filtration $\mathcal{G}$ has the following property:*

$$G^iV_{(n)}G^jV \subset G^{i+j}V \quad \textit{for}\ \ n \leq -1$$

$$G^iV_{(n)}G^jV \subset G^{i+j-1}V \quad \textit{for}\ \ n \geq 0$$

The proof is left as an exercise. An interested reader may find the proof of this fact in a more general setting in [R] .

Lemma 3.15. *Both $\mathcal{F}$ and $\mathcal{G}$ induce the same filtration on $Zhu(V)$.*

Proof. We need to show $F^iV \subset G^iV + O(V)$ and $G^iV \subset F^iV + O(V)$.

We have $F^iV \subset G^iV$ since a monomial of conformal weight less than or equal to i has at most i vector field operators in its formula.

Let us show $G^iV \subset F^iV + O(V)$. Any element of V is a sum of monomials of the form $a_{(-1)}x^1_{(n_1)}\dots x^s_{(n_s)}|0\rangle$ with $a \in V_0$, $\{x^i\}_{1\leq i\leq s} \subset V_1$.

Let $v \in G^iV$ be such a monomial. If v contains $x_{(n)}$ with $x \in \Omega$, then it is in $O'(V)$. Otherwise $s = i$ and, using the proof of Lemma 3.12 one can show that v is equal to $a_{(-1)}x^1_{(-1)}\dots x^s_{(-1)}|0\rangle$ modulo $O'(V)$. This monomial has conformal weight i, so $v \in F^iV + O(V)$. □

The enveloping algebra U_AT is naturally a filtered algebra by (images of) $T^{\otimes n}$, $n \geq 0$. We have

Lemma 3.16. *The map $\alpha : U_AT \to Zhu(V)$ of Lemma 3.7 is a morphism of filtered algebras.*

Proof. Let $F^i = F^i U_A T$ be the i-th filtration subspace of $U_A T$. Clearly, $F^0 = A$, $F^1 = T$ and $F^k \subset (F^1)^k$. Since α is a homomorphism and $\alpha(F^1) = \alpha(T) \subset G^1 Zhu(V)$ we have $\alpha(F^k) \subset (G^1 Zhu(V))^k \subset G^k Zhu(V)$ $\square$

3.3.4. *The Zhu algebra of enveloping algebras. Proof of Theorem 3.1.* Let V be a vertex algebra. Recall that associated to V is a vertex algebroid $\mathcal{A}_V = (A, T, \Omega, \dots)$. In the subsection 3.2.1 we defined the map of associative algebras $\alpha : U_A T \to Zhu(V)$. Theorem 3.1 states that α is surjective when V is generated by $V_0 + V_1$ and it is an isomorphism when V is a vertex envelope of $V_0 + V_1$ and Ω and T are free A-modules. Now we are ready to complete the proof of Theorem 3.1.

(1) Surjectivity follows from Lemma 3.13.

(2) We will show that the induced map

$$\bar{\alpha} : Sym_A(T) \to \mathrm{gr}\ Zhu(V)$$

is an isomorphism of commutative algebras.

Recall [GMS1] that there are canonical filtrations $\mathcal{H}$ on V and $\mathcal{J}$ on $\mathrm{gr}^{\mathcal{G}}\ V$ such that $\mathrm{gr}^{\mathcal{H}}\ V = \mathrm{gr}^{J}\mathrm{gr}^{\mathcal{G}}\ V$ and a canonical map

$$\beta : Sym_A(\bigoplus_{i\geq 0} T^{(i)} \oplus \bigoplus_{i\geq 0} \Omega^{(i)}) \to \mathrm{gr}^{\mathcal{H}}\ V \tag{3.6}$$

which is an isomorphism of commutative algebras provided that Ω and T are free A-modules.

From the definition of the filtration $\mathcal{H}$ and Corollary 3.11, it follows that $I = \mathrm{Symb}^{\mathcal{H}}\mathrm{O(V)}$ is the ideal in $\mathrm{gr}^{\mathcal{H}}\ V$ generated by symbols of $\partial^{(i)}\omega$, $i \geq 0$ and $\partial^{(j)}\tau$, $i \geq 1$ where $\tau \in V_1$.

Hence we have a map

$$\bar{\beta} : Sym_A T \simeq Sym_A(\bigoplus_{i\geq 0} T^{(i)} \oplus \bigoplus_{i\geq 0} \Omega^{(i)})/\beta^{-1}(I) \to \mathrm{gr}^{\mathcal{H}}\ V/I \tag{3.7}$$

which is a commutative algebra isomorphism.

It remains to notice that $\bar{\beta}$ is a composition of $\bar{\alpha}$ with the natural map $\mathrm{gr}\ Zhu(V) \simeq \mathrm{gr}^{\mathcal{G}}\ V/\mathrm{Symb}\ ^{\mathcal{G}}O(V) \to \mathrm{gr}^{\mathcal{H}}\ V/I$ which implies injectivity of α. $\square$

Remark 3.17. The condition that Ω is a free A-module can be dropped, with a slight change to the proof.

3.4. **The Zhu algebra of a CDO.** In this subsection we apply the results obtained above to the sheaf of chiral differential operators.

If $\mathcal{V}$ is a sheaf of vertex algebras on a variety X, we denote $Zhu(\mathcal{V})$ the sheaf associated to the presheaf $U \mapsto Zhu(\mathcal{V}(U))$.

The Theorem 3.1 has the following corollary:

Corollary 3.18. *Let $\mathcal{D}^{ch}_X$ be a CDO on X. Then*

$$Zhu(\mathcal{D}^{ch}_X) \simeq \mathcal{D}_X.$$

For $X = \mathbb{A}^n$ this fact was proved in [L].

Proof. First, let us show that for U that is suitable for chiralization, sect. 2.4, we have an isomorphism $Zhu(\mathcal{D}^{ch}_U) \simeq \mathcal{D}_U$.

The algebra $\mathcal{D}^{ch}_X(U)$ is an enveloping algebra of its 1-truncated part, therefore by Theorem 3.1 there is a natural isomorphism $\alpha_U : \mathcal{D}_X(U) = U_{\mathcal{O}_X(U)}\mathcal{T}(U) \to$

$Zhu(\mathcal{D}^{ch}_X(U))$. For $V \subset U$ the isomorphisms α_V are compatible so that we have an isomorphism of sheaves.

To show that $Zhu(\mathcal{D}^{ch}_X)$ is globally isomorphic to $\mathcal{D}_X$ it is enough to notice that we have the embeddings $\mathcal{O}_X = (\mathcal{D}^{ch}_X)_0 \hookrightarrow Zhu(\mathcal{D}^{ch}_X)$ and $(\mathcal{D}^{ch}_X)_1/\Omega^1 \hookrightarrow Zhu(\mathcal{D}^{ch}_X)$ that implies $\mathcal{T}_X \hookrightarrow Zhu(\mathcal{D}^{ch}_X)$. This is due to the fact that transition functions for $\mathcal{D}^{ch}_X$ are constructed in [GMS1] in such a way that $(\mathcal{D}^{ch}_X)_1/\Omega^1$ is equal to $\mathcal{T}_X$. $\square$

3.5. **The Zhu correspondence for modules.**

Theorem 3.19. *([Zhu]) Let M be a graded module over V. Then the top component M_0 is a module over the Zhu algebra $Zhu(V)$. The assignment $M \mapsto M_0$ establishes a 1-1 correspondence betweem isomorphism classes of irreducible graded V-modules and irreducible $Zhu(V)$-modules.*

Theorem 3.1 implies the following result, which was originally proved by other methods in [LiYam].

Corollary 3.20. *With the assumptions of Theorem 3.1 (2), there is a 1-1 correspondence betweem isomorphism classes of irreducible graded V-modules and irreducible U_AT-modules.*

Remark 3.21. It is enough to assume that M is a filtered V-module.

Remark 3.22. Let $V-$Mod denote the category of graded V-modules M. The assignment

$$\Phi : M \mapsto M_0 \tag{3.8}$$

is a functor from the category of graded (resp. filtered) V-modules to the category of $Zhu(V)$-modules, with the obvious action on maps.

3.5.1. *Left adjoint to the functor (3.8).* Rosellen [R] constructs the left adjoint

$$Zhu_V : Zhu(V)-\mathrm{Mod} \to \mathrm{V}-\mathrm{Mod} \tag{3.9}$$

to the functor (3.8) as follows.

To any vertex Lie algebra R one can attach

$$\mathfrak{g}(R) = R[t, t^{-1}]/(\partial a)(n) + na(n-1),$$

a Lie algebra with bracket

$$[a(n), b(m)] = \sum_{j \geq 0} \binom{n}{j} (a_{(j)}b)(n+m-j).$$

Here $a(n) = at^n$, $a \in R$, $n \in \mathbb{Z}$.

If V is a graded vertex algebra, $\mathfrak{g}(V)$ acquires a Lie algebra grading with $a(n)$ sitting in component $n - \Delta_a + 1$. Let us concentrate on the subalgebra $\mathfrak{g}(V)_0$.

There is a surjective linear map $\mathfrak{g}(V)_0 \to Zhu(V)$ given by $a_0 \mapsto [a]$.

Lemma 3.23. *([R], Proposition 6.1.5) The map $a_0 \mapsto [a]$ from $\mathfrak{g}(V)_0 \to Zhu(V)$ is a Lie algebra map.*

If M is a $Zhu(V)$-module, then M is a $\mathfrak{g}(V)_0$-module by pullback (we have a natural Lie algebra map $\mathfrak{g}(V)_0 \to Zhu(V)$).

Moreover, we can extend this to a $\mathfrak{g}(V)_{\geq}$-module structure by setting $\mathfrak{g}(V)_{>}M = 0$. Then

$$\tilde{M} = U(\mathfrak{g}(V)) \otimes_{U(\mathfrak{g}(V)_{\geq})} M \tag{3.10}$$

is a $\mathbb{Z}_{\geq 0}$-graded $\mathfrak{g}(V)$-module.

Let $Q(\tilde{M})$ be the $\mathfrak{g}(V)$-submodule of $\tilde{M}$ generated by coefficients of

$$(a_{(-1)}b)(z)m - :a(z)b(z):m, \quad m \in \tilde{M}/Q(\tilde{M}).$$

Then $\tilde{M}/Q(\tilde{M})$ is a $\mathbb{Z}_{\geq 0}$-graded V-module.

By definition,

$$Zhu_V(M) = \tilde{M}/Q(\tilde{M}) \tag{3.11}$$

4. Universal twisted cdo

4.1. **Truncated de Rham complexes.** Let X be a smooth algebraic variety. For $0 \leq p < q \leq \dim X$ introduce the complexes

$$\Omega_X^{[p,q>} :\ 0 \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots \Omega_X^{q-1} \to \Omega_X^{q,cl} \to 0,$$

$$\Omega_X^{[p} :\ 0 \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots,$$

$$\widetilde{\Omega}_X^{[q} :\ 0 \to \frac{\Omega_X^q}{\Omega_X^{q,cl}} \to \Omega_X^{q+1} \to \cdots,$$

where Ω_X^m stands for the sheaf of m-forms, the differential is assumed to be the de Rham differential, and the grading is shifted so that Ω_X^p is placed in degree 0.

For any complex of sheaves $\mathcal{A}$ over X consider the hypercohomology groups $H^i(X, \mathcal{A})$, for any cover of X, $\mathfrak{U} = \{U_i\}$, the corresponding Cech hypercohomology, $\check{H}^i(\mathfrak{U}, \mathcal{A})$, and finally the Cech hypercohomology $\check{H}^i(X, \mathcal{A}) = \lim_{\to} \check{H}^i(\mathfrak{U}, \mathcal{A})$.

The diagram

$$\Omega_X^{[p,q>} \to \Omega^{[p} \to \widetilde{\Omega}_X^{[q}$$

is an exact triangle. The corresponding long cohomology sequence and the fact that the de Rham cohomology $H^m_{DR}(X, \widetilde{\Omega}_X^{[q}) = 0$ if $0 \leq m \leq q-p$ implies, cf. [GMS2], sect.4.1.1,

Lemma 4.1. *The canonical maps*

$$H^i(X, \Omega^{[p,q>}) \to H^i(X, \Omega^{[p}),\ \check{H}^i(X, \Omega^{[p,q>}) \to \check{H}^i(X, \Omega^{[p})$$

are isomorphisms if $0 \leq i \leq q-p$ *and injections if* $i = q-p+1$.

Corollary 4.2. *The canonical map*

$$\check{H}^i(X, \Omega_X^{[p,q>}) \to H^i(X, \Omega_X^{[p,q>})$$

is an isomorphism if $i \leq q-p$.

Proof. Since $\Omega_X^{[p}$ is an $\mathcal{O}_X$-module, $\check{H}^i(X, \Omega_X^{[p}) \to H^i(X, \Omega_X^{[p})$ is an isomorphism. The map indicated in the lemma factors as follows

$$\check{H}^i(X, \Omega_X^{[p,q>}) \to \check{H}^i(X, \Omega_X^{[p}) \to H^i(X, \Omega_X^{[p}) \to H^i(X, \Omega_X^{[p,q>}),$$

where thanks to Lemma 4.1 all maps are isomorphisms if $i \leq q-p$. $\square$

4.2. **Twisted differential operators.** A sheaf of twisted differential operators (TDO) is a sheaf of filtered $\mathcal{O}_X$-algebras such that the corresponding graded sheaf is (the push-forward of) $\mathcal{O}_{T^*X}$, see [BB2]. The set of isomorphism classes of such sheaves is in 1-1 correspondence with $H^1(X, \Omega_X^{[1,2>})$. Denote by $\mathcal{D}_X^\lambda$ a TDO that corresponds to $\lambda \in H^1(X, \Omega_X^{[1,2>})$. If $\dim H^1(X, \Omega_X^{[1,2>}) < \infty$, then it is easy to construct a universal TDO, that is to say, a family of sheaves with base $H^1(X, \Omega_X^{[1,2>})$ so that the sheaf that corresponds to a point $\lambda \in H^1(X, \Omega_X^{[1,2>})$ is isomorphic to $\mathcal{D}_X^\lambda$. The construction is as follows.

Assume that X is projective. Then, as Lemma 4.1 implies, $\dim H^1(X, \Omega^{[1,2>}) < \infty$. According to Corollary 4.2, we can pick an affine cover $\mathfrak{U}$ so that $\check{H}^1(\mathfrak{U}, \Omega^{[1,2>}) = H^1(X, \Omega^{[1,2>})$.

Let $\{\lambda_k\}$ and $\{\lambda_k^*\}$ be dual bases of $\check{H}^1(\mathfrak{U}, \Omega^{[1,2>})$ and $(\check{H}^1(\mathfrak{U}, \Omega^{[1,2>}))^*$ respectively. We fix a lifting $\sigma : H^1(\mathfrak{U}, \Omega^{[1,2>}) \to Z^1(\mathfrak{U}, \Omega^{[1,2>})$ and identify the former with a subspace of the latter using this lifting. Upon this identification, each λ_k becomes a pair

$$\lambda_k = (\lambda_k^{(1)}, \lambda_k^{(2)}) \in (\prod_{i,j} \Gamma(U_i \cap U_j, \Omega_X^1)) \times (\prod_i \Gamma(U_i, \Omega_X^{2;cl})) \tag{4.1}$$

so that the forms $\lambda_k^{(1)}(U_i \cap U_j) \in \Gamma(U_i \cap U_j, \Omega_X^1)$ and $\lambda_k^{(2)}(U_i) \in \Gamma(U_i, \Omega_X^{2;cl})$ are defined for each k, i, and j. The cocycle condition reads

$$d_{DR}\lambda_k^{(1)} = d_{\check{C}}\lambda_k^{(2)},\ d_{\check{C}}\lambda_k^{(1)} = 0. \tag{4.2}$$

The space $\mathcal{O}_{U_i} \otimes H^1(X, \Omega^{[1,2>})^*$ carries two obvious actions, by $\mathcal{O}_{U_i}$ and $\mathcal{T}_{U_i}$, defined as follows

$$\mathcal{O}_{U_i} \otimes (\mathcal{O}_{U_i} \otimes H^1(X, \Omega^{[1,2>})^*) \to \mathcal{O}_{U_i} \otimes H^1(X, \Omega^{[1,2>})^*,\ f \otimes (g \otimes h) \mapsto fg \otimes h,$$

$$\mathcal{T}_{U_i} \otimes (\mathcal{O}_{U_i} \otimes H^1(X, \Omega^{[1,2>})^*) \to \mathcal{O}_{U_i} \otimes H^1(X, \Omega^{[1,2>})^*,\ \tau \otimes (g \otimes h) \mapsto \tau(g) \otimes h,$$

and the actions are compatible in that $\tau(f \cdot p) = \tau(f) \cdot p + f \cdot \tau(p)$, $\tau \in \mathcal{T}_{U_i}$, $f \in \mathcal{O}_{U_i}$, $p \in \mathcal{O}_{U_i} \otimes H^1(X, \Omega^{[1,2>})^*$.

Consider an abelian extension of $\mathcal{T}_{U_i}$ by $\mathcal{O}_{U_i} \otimes H^1(X, \Omega^{[1,2>})^*$

$$0 \to \mathcal{O}_{U_i} \otimes H^1(X, \Omega^{[1,2>})^* \to \mathcal{T}_{U_i}^{tw} \to \mathcal{T}_{U_i} \to 0 \tag{4.3}$$

defined by the following cocycle

$$\mathcal{T}_{U_i} \ni \xi, \eta \mapsto \sum_k \iota_\xi \iota_\eta \lambda_k^{(2)}(U_i))\lambda_k^*.$$

In other words, let us define the bracket $\widetilde{[.,.]}$ so that

$$\widetilde{[\xi, \eta]} = [\xi, \eta] + \sum_k (\iota_\xi \iota_\eta \lambda_k^{(2)}(U_i))\lambda_k^*, \tag{4.4}$$

for all $\xi, \eta \in \Gamma(U_i, \mathcal{T}_{U_i})$; here $[\xi, \eta]$ is the usual Lie bracket of vector fields.

The fact that each $\lambda_k^{(2)}(U_i)$ is a closed 2-form implies that $\mathcal{T}_{U_i}^{tw}$ is indeed a Lie algebra, in fact a $\mathcal{O}_{U_i}$-Lie algebroid, $\mathcal{T}_{U_i}^{tw} \to \mathcal{T}_{U_i}$ being the anchor map. Let

$$\mathcal{D}_{U_i}^{tw} = U_{\mathcal{O}_{U_i}} \mathcal{T}_{U_i}^{tw},$$

where $U_{\mathcal{O}_{U_i}}$ is the enveloping algebra functor, cf. the end of sect. 2.4.

Define the transition maps $g_{ij} : \mathcal{D}^{tw}_{U_i}|_{U_i\cap U_j} \to \mathcal{D}^{tw}_{U_j}|_{U_i\cap U_j}$ by requiring that

$$(4.5)\quad g_{ij}(\xi) = \xi - \sum_k (\iota_\xi \lambda^{(1)}_k (U_i \cap U_j))\lambda^*_k,\;\; g_{ij}(f) = f,\;\; f \in \mathcal{O}_{U_i} \otimes \mathbb{C}[H^1(X, \Omega^{[1,2>})].$$

The condition $d_{DR}\lambda^{(1)}_k = d_{\check{C}}\lambda^{(2)}_k$ implies that each g_{ij} is an associative algebra homomorphism, and the condition $d_{\check{C}}\lambda^{(1)}_k = 0$ implies that $g_{ij} = g_{ik} \circ g_{kj}$. Denote by $\mathcal{D}^{tw}_X$ the sheaf obtained by gluing the sheaves $\mathcal{D}^{tw}_{U_i}$ over two-fold intersections via the maps g_{ij}.

By construction, $\mathbb{C}[H^1(X, \Omega^{[1,2>})]$ lies in the center of $\Gamma(X, \mathcal{D}^{tw}_X)$, and if we let $\mathfrak{m}_\lambda \in \mathbb{C}[H^1(X, \Omega^{[1,2>})]$ be the maximal ideal defined by $\lambda \in H^1(X, \Omega^{[1,2>})$, then by definition,

$$(4.6)\qquad \mathcal{D}^{tw}_X/\mathfrak{m}_\lambda \mathcal{D}^{tw}_X \text{ is isomorphic to } \mathcal{D}^\lambda_X.$$

It is clear that $\mathcal{D}^{tw}_X$ is independent of the cover $\mathfrak{U}$ and lifting σ, and we call this sheaf **the universal sheaf of twisted differential operators**.

4.3. **Chiral analogue.**

4.3.1. *A universal twisted CDO.* Let $ch_2(X) = 0$ and fix a CDO $\mathcal{D}^{ch}_X$. To each such sheaf we will attach a *universal twisted CDO*, $\mathcal{D}^{ch,tw}_X$, in a manner analogous to that in which we constructed a universal TDO $\mathcal{D}^{tw}_X$ in the previous section. Let us then place ourselves in the situation of the previous section, where we had a fixed affine cover $\mathfrak{U} = \{U_i\}$ of a projective algebraic manifold X, dual bases $\{\lambda_i\} \in H^1(X, \Omega^{[1,2>}_X)$, $\{\lambda^*_i\} \in H^1(X, \Omega^{[1,2>}_X)^*$, and a lifting $H^1(X, \Omega^{[1,2>}_X) \to Z^1(\mathfrak{U}, \Omega^{[1,2>}_X)$.

Assuming, as we may, that each U_i is suitable for chiralization we fix, for each i, an abelian basis $\tau^{(i)}_1, \tau^{(i)}_2, ...$ of $\Gamma(U_i, \mathcal{T}_X)$, and a collection of 3-forms $\alpha^{(i)} \in \Gamma(U_i, \Omega^{3,cl}_X)$, cf. sect. 2.4, Theorem 2.1.

Lemma 4.3. *(a) There is a unique vertex algebroid structure on the sheaf*

$$\mathcal{A}_{U_i} \stackrel{def}{=} \mathcal{O}_{U_i} \oplus \mathcal{T}_{U_i} \oplus \Omega_{U_i} \oplus \left(\oplus_j \mathcal{O}_{U_i} \otimes \mathbb{C}\lambda^*_j\right)$$

so that

(1) $V_0 = \mathcal{O}_{U_i},\; V_1 = \mathcal{T}_{U_i} \oplus \Omega_{U_i} \oplus \left(\oplus_j \mathcal{O}_{U_i} \otimes \mathbb{C}\lambda^*_j\right);$

(2) $\partial : \mathcal{O}_{U_i} \to \Omega_{U_i}$ *is the de Rham differential*;

(3) the pair $(V_{0,(-1)})$ *is* $\mathcal{O}_{U_i}$ *as a commutative associative algebra;*

(4) $f_{(-1)}\omega = f\omega,\; f \in \mathcal{O}_{U_i}, \omega \in \Omega_{U_i};$

(5) $f_{(-1)}\xi = f\xi \text{ mod } \Omega_{U_i},\; f \in \mathcal{O}_{U_i}, \xi \in \mathcal{T}_{U_i};$

(6) $\tau^{(i)}_{l(0)}\tau^{(i)}_m = \iota_{\tau^{(i)}_l}\iota_{\tau^{(i)}_m}\alpha^{(i)} + \sum_k (\iota_{\tau^{(i)}_l}\iota_{\tau^{(i)}_m}\lambda^{(2)}_k(U_i))\lambda^*_k,\; \tau^{(i)}_{l(0)}f = \tau^{(i)}_l(f), f \in \mathcal{O}_{U_i};$

(7) $\tau^{(i)}_{l(1)}\tau^{(i)}_m = 0;$

(8) $\lambda^*_{k(0)}a = \lambda^*_{k(1)}a = 0$ *for any* k, a.

(b) The corresponding Lie algebroid $T = T(\mathcal{A}_{U_i})$ *satisfies,*

$$T = \mathcal{T}^{tw}_{U_i},$$

where $\mathcal{T}^{tw}_{U_i}$ *is the Lie algebroid that was defined in sect. 4.2.*

Proof.

(a) It is clear, cf. sect. 2.4, that there is only one way to extend the indicated operations to the entire $\mathcal{A}_{U_i}$ using the Borcherds identity (2.1). Furthermore, thus obtained operations are all represented by differential operators. In order to verify that these operations satisfy the identities imposed by the definition of a vertex algebroid, let us embed the sheaf in question, $\mathcal{A}_{U_i}$, into its formal completion, $\hat{\mathcal{A}}_{U_i,x}$, at an arbitrary point $x \in U_i$. All operations on $\mathcal{A}_{U_i}$ extend to those on $\hat{\mathcal{A}}_{U_i,x}$. We will, first, prove that $\hat{\mathcal{A}}_{U_i,x}$ with these operations is a vertex algebroid.

Upon passing to this completion each 2-form $\lambda_k^{(2)}(U_i)$ becomes exact, and for each k we obtain μ_k such that $d_{DR}\mu_k = \lambda_k^{(2)}(U_i)$. Now replace each $\tau_l^{(i)}$ with $\tilde{\tau}_l^{(i)} = \tau_l^{(i)} + \sum_k \iota_{\tau_l^{(i)}}\mu_k\lambda_k^*$. It is clear that in terms of this new basis condition (6) of our lemma becomes

$$(6')\ \tilde{\tau}^{(i)}_{l(0)}\tilde{\tau}^{(i)}_m = \iota_{\tau_l^{(i)}}\iota_{\tau_m^{(i)}}\alpha^{(i)}.$$

This means that the subspace spanned over $\mathbb{C}$ by λ_j^*, $1 \le j \le n$, decouples. More precisely, if we let

$$\widetilde{\mathcal{A}}_{U_i,x} = \hat{\mathcal{O}}_{U_i,x}(\oplus_l \mathbb{C}\tilde{\tau}_l^{(i)}) \oplus \hat{\Omega}_{U_i,x},$$

then the fact that $\tilde{\mathcal{A}}_{U_i,x}$ with operations (1–5,6',7,8) is a vertex algebroid becomes one of the main observations of [GMS1], recorded above as Theorem 2.1.

Adjoining the commutative variables λ_j^* is easy. Condition (8) above simply means that, as a space with operations ∂, ${}_{(n)}$, $n = -1, 0, 1$,

$$\hat{\mathcal{A}}_{U_i,x} = \tilde{\mathcal{A}}_{U_i,x} \overset{\bullet}{\otimes} (\mathbb{C} \oplus (\sum_j \mathbb{C}\lambda_j^*)),$$

where the tensor product functor is as in (2.11) and $\mathbb{C} \oplus (\sum_j \mathbb{C}\lambda_j^*)$ is a commutative vertex algebroid from sect. 2.3.3. Since the R.H.S. is a vertex algebroid, so is the L.H.S., $\hat{\mathcal{A}}_{U_i,x}$.

The map $\mathcal{A}_{U_i} \to \hat{\mathcal{A}}_{U_i,x}$ being an injection, the passage to the completion cannot create any new identities; hence the operations initially defined on $\mathcal{A}_{U_i}$ also satisfy the definition of a vertex algebroid.

(b) It was explained in sect. 2.3.1 that, as an $\mathcal{O}_{U_i}$-module,

$$T = \left(\mathcal{T}_{U_i} \oplus \Omega_{U_i} \oplus \left(\oplus_j \mathcal{O}_{U_i} \otimes \mathbb{C}\lambda_j^*\right)\right)/\Omega_{U_i},$$

hence

$$T = \mathcal{T}_{U_i} \oplus \left(\oplus_j \mathcal{O}_{U_i} \otimes \mathbb{C}\lambda_j^*\right),$$

which is precisely $\mathcal{T}_{U_i}^{tw}$. The Lie bracket is defined by ${}_{(0)}$. It remains to notice that upon passing over to the quotient modulo Ω_{U_i}, formula (6) of Lemma 4.3 becomes exactly formula (4.4). $\square$

Define a sheaf of vertex algebras over each U_i by applying the vertex enveloping algebra functor as follows

$$\mathcal{D}_{U_i}^{ch,tw} \overset{\text{def}}{=} U\mathcal{A}_{U_i}. \tag{4.7}$$

These sheaves will serve as local models for the universal twisted CDO we are after.

By construction we have sheaf embeddings

$$\mathcal{O}_{U_i} \oplus \mathcal{T}_{U_i} \oplus (\oplus_j \mathbb{C}\lambda_j^*) \hookrightarrow \mathcal{D}_{U_i}^{ch,tw}.$$

Recall now that we have assumed given a CDO $\mathcal{D}^{ch}_X$. One way to define this sheaf is to introduce the restrictions $\mathcal{D}^{ch}_{U_i}=\mathcal{D}^{ch}_X|_{U_i}$, fix splittings

$$\mathcal{O}_{U_i}\oplus\mathcal{T}_{U_i}\hookrightarrow\mathcal{D}^{ch,tw}_{U_i},$$

and the corresponding transition functions

$$g_{ij}:\mathcal{D}^{ch}_{U_i}|_{U_i\cap U_j}\to\mathcal{D}^{ch}_{U_j}|_{U_i\cap U_j}.$$

Lemma 4.4. *(1) There is a unique vertex algebra isomorphism*

$$g^{tw}_{ij}:\mathcal{D}^{ch,tw}_{U_i}|_{U_i\cap U_j}\to\mathcal{D}^{ch,tw}_{U_j}|_{U_i\cap U_j}$$

such that

$$g^{tw}_{ij}|_{\mathcal{O}_{U_i\cap U_j}}=g_{ij}|_{\mathcal{O}_{U_i\cap U_j}}$$
$$g^{tw}_{ij}(\lambda^*_k)=\lambda^*_k,$$
$$g^{tw}_{ij}(\xi)=g_{ij}(\xi)-\sum_k(\iota_\xi\lambda^{(1)}_k(U_{ij}))\lambda^*_k,\xi\in\mathcal{T}_{U_i\cap U_j}.$$

(2) On triple intersections $U_i\cap U_j\cap U_k$

$$g^{tw}_{ij}=g^{tw}_{kj}\circ g^{tw}_{ik}.$$

Proof.

(1)As it follows from the *Reconstruction theorem*, [K], the fact on which an analogous discussion in [MSV] heavily relies, it is enough to verify the equalities

$$g^{tw}_{ij}(\xi)_{(1)}g^{tw}_{ij}(\eta)=g^{tw}_{ij}(\xi_{(1)}\eta),\ g^{tw}_{ij}(\xi)_{(0)}g^{tw}_{ij}(\eta)=g^{tw}_{ij}(\xi_{(0)}\eta),\ \xi,\eta\in\mathcal{T}_{U_i\cap U_j}.$$

The former is part of the definition of $\mathcal{D}^{ch}_X$ for we have, by definition, $g^{tw}_{ij}(\xi)_{(1)}g^{tw}_{ij}(\eta)=g_{ij}(\xi)_{(1)}g_{ij}(\eta)$ and $g^{tw}_{ij}(\xi_{(1)}\eta)=g_{ij}(\xi_{(1)}\eta)$.

The latter boils down to the purely classical statement that underlies the construction of the twisted differential operators, see sect. 4.2. Note that the deformation of $_{(0)}$ by a function introduced in Lemma 4.3(6) has as a consequence the fact that the "old" transition functions, g_{ij}, are no longer vertex algebra morphisms, the discrepancy being

$$g_{ij}(\xi_{(0)}\eta)-g_{ij}(\xi)_{(0)}g_{ij}(\eta)=\sum_k\iota_\xi\iota_\eta(\lambda^{(2)}_k(U_i)-\lambda^{(2)}_k(U_j))\lambda^*_k.$$

This discrepancy is taken care of by the passage from g_{ij} to g^{tw}_{ij}. Indeed, since by definition

$$g_{ij}(\xi)_{(0)}\sum_k(\iota_\eta\lambda^{(1)}_k(U_{ij}))\lambda^*_k=-(\sum_k(\iota_\eta\lambda^{(1)}_k(U_{ij}))\lambda^*_k)_{(0)}g_{ij}(\xi)=\sum_k\xi(\iota_\eta\lambda^{(1)}_k(U_{ij}))\lambda^*_k,$$

we have

$$g^{tw}_{ij}(\xi_{(0)}\eta)=g_{ij}(\xi_{(0)}\eta)-\sum_k\iota_{[\xi,\eta]}\lambda^{(1)}_k(U_{ij})\lambda^*_k;$$

$$g^{tw}_{ij}(\xi)_{(0)}g^{tw}_{ij}(\eta)=g_{ij}(\xi)_{(0)}g_{ij}(\eta)-\sum_k\xi((\iota_\eta\lambda^{(1)}_k(U_{ij}))\lambda^*_k+\sum_k\eta((\iota_\xi\lambda^{(1)}_k(U_{ij}))\lambda^*_k.$$

Subtracting the latter from the former we obtain

$$g^{tw}_{ij}(\xi_{(0)}\eta)-g^{tw}_{ij}(\xi)_{(0)}g^{tw}_{ij}(\eta)=$$
$$\sum_k(\iota_\xi\iota_\eta(\lambda^{(2)}_k(U_i)-\lambda^{(2)}_k(U_j))+\sum_k(\iota_\eta\iota_\xi d_{DR}\lambda^{(1)}_k(U_{ij}))\lambda^*_k=$$

$$-\sum_k (d_{\check{C}}(\lambda_k^{(2)}(U_{ij}))|_{\xi,\eta} - d_{DR}\lambda_k^{(1)}(U_{ij})|_{\xi,\eta})\lambda_k^*$$

which vanishes by virtue of the first part of the cocycle condition (4.2).

(2) This is also a statement about twisted differential operators: we have over $U_i \cap U_j \cap U_k$

$$g_{ij}^{tw}(\xi) - g_{kj}^{tw} \circ g_{ik}^{tw}(\xi) = g_{ij}(\xi) - g_{kj} \circ g_{ik}(\xi) - \sum_k (d_{\check{C}}\lambda_k^{(1)}|_{\xi})\lambda_k^*,$$

which vanishes by virtue of the second part of the cocycle condition (4.2). $\square$

Lemma 4.4 implies the following

Theorem-Definition 4.5. *Given a projective algebraic manifold X and a CDO $\mathcal{D}_X^{ch}$, there is a unique sheaf of vertex algebras, to be denoted $\mathcal{D}_X^{ch,tw}$ and called* **a universal sheaf of twisted chiral differential operators** *(TCDO), such that*

$$\mathcal{D}_X^{ch,tw}|_{U_i} = \mathcal{D}_{U_i}^{ch,tw},$$

and the canonical isomorphisms

$$\mathcal{D}_{U_i}^{ch,tw}|_{U_i \cap U_j} \to \mathcal{D}_{U_j}^{ch,tw}|_{U_i \cap U_j}$$

are g_{ij}^{tw}.

Indeed, the assumptions of the theorem require that $\mathcal{D}_X^{ch,tw}$ be obtained by gluing the pieces $\mathcal{D}_{U_i}^{ch,tw}$ via g_{ij}^{tw}, and the gluing is made sense of by Lemma 4.4. $\square$

Let H_X be the commutative vertex algebra of differential polynomials on $H^1(X, \Omega_X^{[1,2>})$, cf. sect. 2.2.2. As a commutative algebra

$$H_X = \mathbb{C}[\partial^j \lambda_k^*; j \geq 0, 1 \leq k \leq \dim H^1(X, \Omega_X^{[1,2>})],$$

and the canonical derivation is ∂.

Denote by $\underline{\mathrm{H}}_X$ the constant sheaf over X with $\underline{\mathrm{H}}_X(U) = H_X$ for nonempty U.

It is clear that if we let operations $_{(0)}, _{(1)} = 0$, then $\mathbb{C} \oplus (\oplus_j \mathbb{C}\lambda_j^*)$ is a vertex algebroid and that $U(\mathbb{C} \oplus (\oplus_j \mathbb{C}\lambda_j^*) = H_X$. Now the embeddings

$$\underline{\mathrm{H}}_X \hookrightarrow Z(\mathcal{D}_X^{ch,tw}),\ H_X \hookrightarrow Z(\Gamma(X, \mathcal{D}_X^{ch,tw})) \tag{4.8}$$

follow from the constructions at once; here for any vertex algebra V, $Z(V)$ stands for its center, that is to say, $Z(V) = \{v \in V \text{ s.t. } v_{(n)}V = 0 \text{ for all } n \geq 0\}$.

4.3.2. *Locally trivial and other versions of twisted CDOs.* To begin with, note that the requirement that X be projective was needed above only to ensure that $H^1(X, \Omega_X^{[1,2>})$ is finite-dimensional. In the infinite-dimensional situation one has to work with completions, which may well be possible but not attractive.

On the other hand, for any X and a fixed cover $\mathfrak{U}$ one can repeat all of the above constructions and obtain sheaves $\mathcal{D}_{X,\mathfrak{U}}^{tw}$ and $\mathcal{D}_{X,\mathfrak{U}}^{ch,tw}$. Such sheaves will not be universal in general and will explicitly depend on the choice of $\mathfrak{U}$.

Yet another version of our construction will give us locally trivial twisted sheaves of chiral differential operators.

There is an embedding

$$H^1(X, \Omega_X^{1,cl}) \hookrightarrow H^1(X, \Omega_X^{[1,2>}) \tag{4.9}$$

The space $H^1(X,\Omega_X^{1,cl})$ classifies *locally trivial* twisted differential operators, those that are locally isomorphic to $\mathcal{D}_X$. Thus for each $\lambda\in H^1(X,\Omega_X^{1,cl})$, there is a unique up to isomorphism TDO $\overset{\circ}{\mathcal{D}}{}_X^{\lambda}$ such that for each sufficiently small open $U\subset X$, $\overset{\circ}{\mathcal{D}}{}_X^{\lambda}|_U$ is isomorphic to $\mathcal{D}_U$. Let us see what this means at the level of the universal TDO.

In terms of Cech cocycles the image of embedding (4.9) is described by those $(\lambda^{(1)},\lambda^{(2)})$, see (4.1), where $\lambda^{(2)}=0$, and this forces $\lambda^{(1)}$ to be closed. Picking a collection of such cocycles that represent a basis of $H^1(X,\Omega_X^{1,cl})$ we can repeat the constructions of sections 4.2 and 4.3.1 to obtain sheaves $\overset{\circ}{\mathcal{D}}{}_X^{tw}$ and $\overset{\circ}{\mathcal{D}}{}_X^{ch,tw}$. The former is glued of the pieces isomorphic to $\mathcal{D}_{U_i}\otimes\mathbb{C}[H^1(X,\Omega_X^{1,cl})]$ as associative algebras (this is a locally trivial property, it is due to the vanishing of $\lambda^{(2)}$), the transition functions being defined as in (4.5). The latter is defined likewise by gluing pieces isomorphic (as vertex algebras) to $\mathcal{D}^{ch}_{U_i}\otimes H_X$ with transition functions as in Lemma 4.4; here H_X is the vertex algebra of differential polynomials on $H^1(X,\Omega_X^{1,cl})$. We will call the sheaves $\overset{\circ}{\mathcal{D}}{}_X^{tw}$ and $\overset{\circ}{\mathcal{D}}{}_X^{ch,tw}$ the *universal locally trivial sheaves of twisted (chiral resp.) differential operators.*

4.3.3. *Example: flag manifolds.* Let us see what our constructions give us if $X=\mathbb{P}^1$. We have $\mathbb{P}^1=\mathbb{C}_0\cup\mathbb{C}_\infty$, a cover $\mathfrak{U}=\{\mathbb{C}_0,\mathbb{C}_\infty\}$, where $\mathbb{C}_0$ is $\mathbb{C}$ with coordinate x, $\mathbb{C}_\infty$ is $\mathbb{C}$ with coordinate y, with the transition function $x\mapsto 1/y$ over $\mathbb{C}^*=\mathbb{C}_0\cap\mathbb{C}_\infty$.

Defined over $\mathbb{C}_0$ and $\mathbb{C}_\infty$ are the standard CDOs, $\mathcal{D}^{ch}_{\mathbb{C}_0}$ and $\mathcal{D}^{ch}_{\mathbb{C}_\infty}$. The spaces of global sections of these sheaves are polynomials in $\partial^n(x)$, $\partial^n(\partial_x)$ (or $\partial^n(y)$, $\partial^n(\partial_y)$ in the latter case), where ∂ is the translation operator, so that, cf. sect. 2.4,

$$(\partial_x)_{(0)}x=(\partial_y)_{(0)}y=1.$$

There is a unique up to isomorphism CDO on $\mathbb{P}^1$, $\mathcal{D}^{ch}_{\mathbb{P}^1}$; it is defined by gluing $\mathcal{D}^{ch}_{\mathbb{C}_0}$ and $\mathcal{D}^{ch}_{\mathbb{C}_\infty}$ over $\mathbb{C}^*$ as follows [MSV]:

$$x\mapsto 1/y,\ \partial_x\mapsto(-\partial_y)_{(-1)}(y^2)-2\partial(x). \tag{4.10}$$

The canonical Lie algebra morphism

$$sl_2\to\Gamma(\mathbb{P}^1,\mathcal{T}_{\mathbb{P}^1}), \tag{4.11}$$

where

$$e\mapsto\partial_x,\quad h\mapsto -2x\partial_x,\quad f\mapsto -x^2\partial_x, \tag{4.12}$$

e,h,f being the standard generators of sl_2, can be lifted to a vertex algebra morphism

$$V_{-2}(sl_2)\to\Gamma(\mathbb{P}^1,\mathcal{D}^{ch}_{\mathbb{P}^1}), \tag{4.13}$$

where

$$e_{(-1)}|0\rangle\mapsto\partial_x, h(-1)|0\rangle\mapsto -2(\partial_x)_{(-1)}x, f_{(-1)}|0\rangle\mapsto -(\partial_x)_{(-1)}x^2-2\partial(x). \tag{4.14}$$

The twisted version of all of this is as follows.

Since $\dim\mathbb{P}^1=1$,

$$H^1(\mathbb{P}^1,\Omega^1_{\mathbb{P}^1}\to\Omega^{2,cl}_{\mathbb{P}^1})=\Omega^1_{\mathbb{P}^1}=\Omega^{1,cl}_{\mathbb{P}^1},$$

so all twisted CDO on $\mathbb{P}^1$ are locally trivial. Furthermore, $H^1(\mathbb{P}^1,\Omega^{1,cl}_{\mathbb{P}^1})=\mathbb{C}$ and is spanned by the cocycle $\mathbb{C}_0\cap\mathbb{C}_\infty\mapsto dx/x$, the Chern class of Serre's sheaf $\mathcal{O}(1)$.

We have $H_{\mathbb{P}^1} = \mathbb{C}[\lambda^*, \partial(\lambda^*), ...]$. Let $\mathcal{D}^{ch,tw}_{\mathbb{C}_0} = \mathcal{D}^{ch}_{\mathbb{C}_0} \otimes H_{\mathbb{P}^1}$, $\mathcal{D}^{ch,tw}_{\mathbb{C}_\infty} = \mathcal{D}^{ch}_{\mathbb{C}_\infty} \otimes H_{\mathbb{P}^1}$ and define $\mathcal{D}^{ch,tw}_{\mathbb{P}^1}$ by gluing $\mathcal{D}^{ch,tw}_{\mathbb{C}_0}$ onto $\mathcal{D}^{ch,tw}_{\mathbb{C}_\infty}$ via

$$\lambda^* \mapsto \lambda^*,\ x \mapsto 1/y,\ \partial_x \mapsto -(\partial_y)_{(-1)} y^2 - 2\partial(y) + y_{(-1)}\lambda^*. \tag{4.15}$$

Morphism (4.13) "deforms" to

$$V_{-2}(sl_2) \to \Gamma(\mathbb{P}^1, \mathcal{D}^{ch,tw}_{\mathbb{P}^1}), \tag{4.16}$$

(4.17)
$$e_{(-1)}|0\rangle \mapsto \partial_x, h(-1)|0\rangle \mapsto -2(\partial_x)_{(-1)}x + \lambda^*, f_{(-1)}|0\rangle \mapsto -(\partial_x)_{(-1)}x^2 - 2\partial(x) + x_{(-1)}\lambda^*.$$

Furthermore, consider $T = e_{(-1)}f_{(-1)} + f_{(-1)}e_{(-1)} + 1/2h_{(-1)}h \in V_{-2}(sl_2)$. It is known that $T \in \mathfrak{z}(V_{-2}(sl_2))$, the center of $V_{-2}(sl_2)$, and in fact, the center $\mathfrak{z}(V_{-2}(sl_2))$ equals the commutative vertex algebra of differential polynomials in T. The formulas above show

$$T \mapsto \frac{1}{2}\lambda^*_{(-1)}\lambda^* - \partial(\lambda^*) \in H_{\mathbb{P}^1}. \tag{4.18}$$

All of the above is easily verified by direct computations, cf. [MSV]. The higher rank analogue is less explicit but valid nevertheless.

Let G be a simple complex Lie group, $B \subset G$ a Borel subgroup, $X = G/B$, the flag manifold, $\mathfrak{g} = \text{Lie } G$ the corresponding Lie algebra, $\mathfrak{h}$ a Cartan subalgebra. One has a sequence of maps

$$\mathfrak{h}^* \to H^1(X, \Omega^{1,cl}_X) \to H^1(X, \Omega^1_X \to \Omega^{2,cl}_X). \tag{4.19}$$

The leftmost map attaches to an integral weight $\lambda \in P \subset \mathfrak{h}^*$ the Chern class of the G-equivariant line bundle $\mathcal{L}_\lambda = G \times_B \mathbb{C}_\lambda$, and then extends thus defined map $P \to H^1(X, \Omega^{1,cl}_X)$ to $\mathfrak{h}^*$ by linearity. The rightmost one is engendered by the standard spectral sequence converging to hypercohomology. It is easy to verify that both these maps are isomorphisms. Therefore,

$$\mathfrak{h}^* \xrightarrow{\sim} H^1(X, \Omega^{1,cl}_X) \xrightarrow{\sim} H^1(X, \Omega^1_X \to \Omega^{2,cl}_X), \tag{4.20}$$

and each twisted CDO on X is locally trivial.

Note that $\mathcal{L}_\lambda$ being G-equivariant, there arises a map from $U\mathfrak{g}$ to the algebra of differential operators acting on $\mathcal{L}_\lambda$ or, equivalently, [BB2],

$$U\mathfrak{g} \to \mathcal{D}^\lambda_X.$$

A moment's thought shows that this map is a polynomial in λ; hence it defines a universal map

$$U\mathfrak{g} \to \mathcal{D}^{tw}_X. \tag{4.21}$$

Constructed in [MSV] is a (unique up to isomorphism [GMS2]) CDO $\mathcal{D}^{ch}_X$. We arrive at the universal twisted CDO $\mathcal{D}^{ch,tw}_X$ locally isomorphic to $\mathcal{D}^{ch}_U \otimes H_X$, where H_X is the vertex algebra of differential polynomials on $\mathfrak{h}^*$.

Constructed in [MSV] – or rather in [FF1], see also [F1] and [GMS2] for an alternative approach – is a vertex algebra morphism

$$V_{-h^\vee}(\mathfrak{g}) \to \Gamma(X, \mathcal{D}^{ch}_X). \tag{4.22}$$

Furthermore, it is an important result of Feigin and Frenkel [FF2], see also an excellent presentation in [F1], that $V_{-h^\vee}(\mathfrak{g})$ possesses a non-trivial center, $\mathfrak{z}(V_{-h^\vee}(\mathfrak{g}))$,

which, as a vertex algebra, isomorphic to the algebra of differential polynomials in $\mathrm{rk}\mathfrak{g}$ variables.

Lemma 4.6. *Morphism (4.22) "deforms" to*

$$\rho: \ V_{-h^{\vee}}(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X^{ch,tw})$$

and

$$\rho(\mathfrak{z}(V_{-h^{\vee}}(\mathfrak{g}))) \subset H_X.$$

Sketch of Proof. We will be brief, because this is one of those proofs that the reader may find easier to find on his own than to read somebody else's explanations. For each $x \in \mathfrak{g}$, $\rho(x)$ can be written, schematically, as follows

$$\rho(x) = (classical) + (chiral) + (classical)_\lambda,$$

where $(classical)$ are those terms that appear in the image of the canonical map $U\mathfrak{g} \to \mathcal{D}_X$, $(classical) + (classical)_\lambda$ are those that appear in the image of the Beilinson-Bernstein map (4.21), and $(chiral)$ is the rest; note that equivalently $(classical) + (chiral)$ is the image of map (4.22).

We have to verify that $\rho(x)_{(1)}\rho(y) = -h^\vee < x, y >$ and $\rho(x)_{(0)}\rho(y) = \rho([x, y])$. Only terms $(classical) + (chiral)$ contribute to the former; that their contribution is as needed is the content of assertion (4.22). Given the former, the latter becomes precisely the classical construction of the morphism $U\mathfrak{g} \to \mathcal{D}_X^{tw}$.

The assertion on the image of the center was actually verified in [FF2, F1]. Indeed, since H_X is the space of global sections of the constant sheaf $\underline{\mathrm{H}}_X$, it is enough to verify the assertion for the composition of ρ with the embedding of $\Gamma(X, \mathcal{D}_X^{ch,tw})$ in $\Gamma(X_e, \mathcal{D}_X^{ch,tw})$, where $X_e \subset X$ is the big cell. The space $\Gamma(X_e, \mathcal{D}_X^{ch,tw})$ is a Wakimoto module, and it is the properties of thus defined morphism from $V_{-h^{\vee}}(\mathfrak{g})$ to the Wakimoto module that were studied in [FF2, F1]. $\square$

4.4. **The Zhu algebra of $\mathcal{D}_X^{ch,tw}$.** Now we compute the Zhu algebra for the sheaf $\mathcal{D}_X^{ch,tw}$. We show that the obtained sheaf is the universal sheaf $\mathcal{D}_X^{tw}$ of twisted differential operators on X.

Theorem 4.7. *Suppose $\mathcal{D}_X^{ch}$ is a CDO on X. Let $\mathcal{D}_X^{ch,tw}$ be the corresponding twisted sheaf. Then*

$$Zhu(\mathcal{D}_X^{ch,tw}) = \mathcal{D}_X^{tw}. \tag{4.23}$$

Likewise

$$Zhu(\overset{\circ}{\mathcal{D}}{}_X^{ch,tw}) = \overset{\circ}{\mathcal{D}}{}_X^{tw}. \tag{4.24}$$

Proof. Let us compute $Zhu(\mathcal{D}_X^{ch,tw}(U))$ for $U_i \in \mathfrak{U}$.

By definition, see (4.7), we have $\mathcal{D}_{U_i}^{ch,tw} = U\mathcal{A}_{U_i}$.

Lemma 4.3 (b) says that the corresponding Lie algebroid is $\mathcal{T}_{U_i}^{tw}$. Now Theorem 3.1 implies that $Zhu(\mathcal{D}_{U_i}^{ch,tw}) = U_{\mathcal{O}_{U_i}}\mathcal{T}_{U_i}^{tw}$. The latter by definition equals $\mathcal{D}_{U_i}^{tw}$.

It remains to show that the transition functions are as claimed, and this is obvious.

Literally the same proof applies to the locally trivial TCDO $\overset{\circ}{\mathcal{D}}{}_X^{ch,tw}$. $\square$

5. Modules over a universal twisted CDO

5.1. **The main result.** We will call a sheaf of vector spaces $\mathcal{M}$ a $\mathcal{D}_X^{ch,tw}$-*module* if

(1) for each open $U\subset X$ is a $\Gamma(U,\mathcal{D}_X^{ch,tw})$-module;

(2) the restriction morphisms $\Gamma(U,\mathcal{M})\to\Gamma(V,\mathcal{M})$, $V\subset U$, are are $\Gamma(U,\mathcal{D}_X^{ch,tw})$-module morphisms, where the $\Gamma(U,\mathcal{D}_X^{ch,tw})$-module structure on $\Gamma(V,\mathcal{M})$ is that of the pull-back w.r.t. to the restriction map $\Gamma(U,\mathcal{D}_X^{tw})\to\Gamma(V,\mathcal{D}_X^{tw})$;

(3) $\mathcal{M}$ is generated by a subsheaf $\mathcal{M}_0$ such that for each open $U\subset X$

$$v_n\Gamma(U,\mathcal{M}_0)=0\quad\text{for } v\in\Gamma(U,\mathcal{D}_X^{ch,tw}),\ n>0, \tag{5.1}$$

$$v_0\Gamma(U,\mathcal{M}_0)\subset\Gamma(U,\mathcal{M}_0)\quad\text{for } v\in\Gamma(U,\mathcal{D}_X^{ch,tw}). \tag{5.2}$$

Remark 5.1. Note that condition (3) implies a $\mathcal{D}_X^{ch,tw}$-module $\mathcal{M}$ is filtered, i.e.

$$\{0\}\subset\mathcal{M}_0\subset\mathcal{M}_1\subset\cdots,\ \mathcal{M}=\cup_{n=0}^{\infty}\mathcal{M}_n,\ \text{with } \mathcal{M}_j\stackrel{\text{def}}{=}\sum_{i=0}^{j}(\mathcal{D}_X^{ch,tw})_i\mathcal{M}_0, \tag{5.3}$$

and this filtration is compatible with the conformal weight grading of $\mathcal{D}_X^{ch,tw}$ in that

$$((\mathcal{D}_X^{ch,tw})_j)_{(l)}\mathcal{M}_n\subset\mathcal{M}_{j+n-l-1}. \tag{5.4}$$

Denote by $Mod-\mathcal{D}_X^{ch,tw}$ the category of $\mathcal{D}_X^{ch,tw}$-modules.

Precisely the same definition can be made in the case of a locally trivial TCDO, $\overset{\circ}{\mathcal{D}}{}_X^{ch,tw}$, see sect. 4.3.2 and we obtain the category $Mod-\overset{\circ}{\mathcal{D}}{}_X^{ch,tw}$.

Recall that $\mathcal{D}_X^{ch,tw}$ contains a huge center $H_X\subset Z(\Gamma(X,\mathcal{D}_X^{ch,tw}))$, see (4.8). Since the vertex algebra H_X is commutative, its irreducibles are all 1-dimensional, characters in other words, and are in 1-1 correspondence with the algebra of Laurent series with values in $H^1(X,\Omega_X^{[1,2>})$. Specifically, if $\chi(z)\in H^1(X,\Omega_X^{[1,2>})((z))$, then the character $\mathbb{C}_\chi$ is a 1-dimensional H_X-module defined by, cf (2.2, 2.3)

$$\chi: H_X\to Fields(\mathbb{C}_\chi),\ \chi(\lambda)(z)=\lambda(\chi(z)). \tag{5.5}$$

For example, if $\lambda\in H^1(X,\Omega_X^{[1,2>})^*$, thus λ is a linear function, and $\chi(z)=\sum_n\chi_n z^{-n-1}$, then

$$\chi(\lambda)(z)=\sum_{n\in ZZ}\lambda(\chi_n)z^{-n-1}\text{ or }\chi(\lambda)_{(n)}=\lambda(\chi_n).$$

Denote by $Mod_\chi-\mathcal{D}_X^{ch,tw}$ the full subcategory of $Mod-\mathcal{D}_X^{ch,tw}$ consisting of those $\mathcal{D}_X^{ch,tw}$-modules where H_X acts according to the character χ.

We will say that a character $\chi(z)\in H^1(X,\Omega_X^{[1,2>})((z))$ has *regular singularity* if $\chi(z)=\chi_0z^{-1}+\chi_{-1}+\chi_{(-2)}z+\cdots$.

If $\mathcal{M}\in Mod-\mathcal{D}_X^{ch,tw}$, then according to Theorem 4.7, $\mathcal{M}_0$ is a $\mathcal{D}_X^{tw}$-module (even though $\mathcal{M}$ is filtered and not graded, the fact that the Zhu algebra acts on the top filtered component remains obviously true). If, in addition, $\mathcal{M}\in Mod_\chi-\mathcal{D}_X^{ch,tw}$ for some $\chi(z)$ with regular singularity, then the action of $\mathcal{D}_X^{tw}$ factors through the projection $\mathcal{D}_X^{tw}\to\mathcal{D}_X^{\chi_0}$, see (4.6), and we obtain a functor

$$\Phi:\ Mod_\chi-\mathcal{D}_X^{ch,tw}\to Mod-\mathcal{D}_X^{\chi_0}, \tag{5.6}$$

where $Mod-\mathcal{D}_X^{\chi_0}$ stands for the category of $\mathcal{D}_X^{\chi_0}$-modules.

The locally trivial version

$$\Phi:\ Mod_\chi - \overset{\circ}{\mathcal{D}}_X^{ch,tw} \to Mod\!- \overset{\circ}{\mathcal{D}}_X^{\chi_0} \tag{5.7}$$

is immediate.

The purpose of this section is to prove the following theorem.

Theorem 5.2. *(1) The category $Mod_\chi - \mathcal{D}_X^{ch,tw}$ consists of only one object, $\{0\}$, unless $\chi(z)$ has regular singularity.*

(2) If $\chi(z)$ has regular singularity, then functor (5.6) is exact and establishes an equivalence of categories.

(3) Assertions (1,2) remain valid upon replacing $\mathcal{D}_X^{ch,tw}$ with $\overset{\circ}{\mathcal{D}}_X^{ch,tw}$.

5.2. **Proof of Theorem 5.2.** Assertion (1) is obvious for if $\chi(z)$ has an irregular singularity, then condition (3) of the definition of a $\mathcal{D}_X^{ch,tw}$ is violated for the subsheaf $\underline{\mathrm{H}}_X$.

In order to prove assertion (2) we will construct the left adjoint to (5.6) and show that it is a quasi-inverse of (5.6).

5.2.1. *The left adjoint to (5.6).* We begin by constructing the left adjoint functor locally.

Denote $Mod_\chi\!-\!\mathcal{D}_X^{ch,tw}(U)$ the category of filtered $\mathcal{D}_X^{ch,tw}(U)$-modules defined by analogy with $Mod_\chi\!-\!\mathcal{D}_X^{ch,tw}$.

The functor $M \mapsto M_0$ from $Mod_\chi\!-\!\mathcal{D}_X^{ch,tw}(U)$ to $Mod\!-\!\mathcal{D}_X^{\chi_0}(U)$ admits a left adjoint Zhu_χ. It is constructed as follows.

Let F be a $\Gamma(U,\mathcal{D}_X^{\chi_0})$-module. In particular, it is a $\Gamma(U,\mathcal{D}_X^{tw})$-module, by pullback; therefore we may apply the functor Zhu_V, see section 3.5.1, to it. We define

$$\tilde{F} = Zhu_V(F),\quad V = \mathcal{D}_X^{ch,tw}(U). \tag{5.8}$$

For a graded $\Gamma(U,\mathcal{D}_X^{ch,tw})$-module N denote $K_\chi(N)$ to be the subspace spanned by vectors of the form

$$(\lambda_k^*)_{(n)}m - \lambda_k^*(\chi_n)m,$$

where $m\in N$, $n\leq -1$, $1\leq k\leq \dim H^1(X,\Omega_X^{[1,2>})$. It is easy to see that $K_\chi(N)$ is a submodule of N.

Define

$$Zhu_\chi(F) = \tilde{F}/K_\chi(\tilde{F}). \tag{5.9}$$

The character $\chi(z)$ having regular singularity, conditions (5.1,5.2) are satisfied; by construction of the Zhu_V-functor, sect. 3.5.1, Condition (3) of a $\Gamma(U,\mathcal{D}_X^{ch,tw})$-module is satisfied.

Any $\mathcal{D}^{\chi_0}(U)$-module map $f: F\to F'$ extends uniquely to a map $Zhu(f): Zhu_\chi(F)\to Zhu_\chi(F')$. Therefore, the functor Zhu_χ is the left adjoint to the functor $M\to M_0$.

Now we proceed to define a sheaf version of Zhu_χ.

If $\mathcal{M}$ is a $\mathcal{D}_X^{\chi_0}$-module, let us denote by $\mathcal{Z}hu_\chi(\mathcal{M})$ the sheaf associated to the presheaf

$$U\mapsto Zhu_{\chi,U}\mathcal{M}(U) \tag{5.10}$$

with restriction maps extended uniquely from those of $\mathcal{M}$. It is clear that $\mathcal{Z}hu_\chi(\mathcal{M})$ is a sheaf of $\mathcal{D}_X^{ch,tw}$-modules. Since maps extend uniquely, this extends to a functor

$$\mathcal{Z}hu_\chi : Mod\text{–}\mathcal{D}_{\mathrm{X}}^{\chi_0} \to \mathrm{Mod}_\chi\text{–}\mathcal{D}_{\mathrm{X}}^{\mathrm{ch,tw}} \tag{5.11}$$

left adjoint to the functor 5.6.

5.2.2. *The quasi-inverse property.* We have to show the following two functor isomorphisms

$$\Phi \circ \mathcal{Z}hu_\chi \xrightarrow{\sim} \mathrm{Id}_{Mod-\mathcal{D}_{\mathrm{X}}^{\chi_0}}, \tag{5.12}$$

$$\mathcal{Z}hu_\chi \circ \Phi \xrightarrow{\sim} \mathrm{Id}_{Mod_\chi-\mathcal{D}_{\mathrm{X}}^{\mathrm{ch,tw}}}, \tag{5.13}$$

The first is obviously true, because by construction the functors are actually equal: $\Phi \circ \mathcal{Z}hu_\chi = \mathrm{Id}_{Mod-\mathcal{D}_{\mathrm{X}}^{\chi_0}}$. Let us now prove (5.13).

Let $U \subset X$ be a suitable for chiralization open subset of X, $A = \Gamma(U, \mathcal{O}_X)$, $\{\partial^i\}$ be an abelian basis for A-module $\Gamma(U, \mathcal{T}_X)$, and $\{\omega_i\}$ the dual basis of $\Gamma(U, \Omega^1_X)$.

Let $V = \Gamma(U, \mathcal{D}_X^{ch,tw})$, M a V-module

Fix a splitting $s : \mathcal{T} \to V_1$. We will identify $\mathcal{T}_U$ and $s(\mathcal{T}_U) \subset V_{\leq 1}$.

We will denote the kth mode of $s(\partial^i)$ (resp. ω_i) by $\partial_{i,k}$ (resp. $\omega_{i,k}$.)

Let P denote the polynomial algebra in variables $\left\{D^i_{-n}, \Omega^i_{-n},\ n > 0,\ 1 \leq i \leq \dim X\right\}$

Define the map $a : P \to \mathrm{End}\, M$, $a(D^i_{-n}) = \partial_{i,-n}$, $a(\Omega^i_{-n}) = \omega_{i,-n}$.

Choose any total order $\succeq$ on the set of variables that satisfies $D^i_{-n} \succeq \Omega^j_{-m} \succeq 1$ for all $m > 0$, $n > 0$, i, j, and $A_{-n} \succeq B_{-m}$ if $n > m$ for A and B being either D^i or Ω^j.

Define $\Psi : P \otimes M_0 \to M$ as follows:

$$x^1 x^2 \ldots x^k \otimes m \quad \mapsto \quad a(x^1)a(x^2)\ldots a(x^k)m \tag{5.14}$$

where $x^1 \succeq x^2 \succeq \cdots \succeq x^k$; for $k = 0$ set Ψ to be the identity map of M_0.

Lemma 5.3. *Suppose M is a filtered $\Gamma(U, \mathcal{D}_X^{ch,tw})$-module generated by M_0, on which H_X acts via the character $\chi(z) \in H^1(X, \Omega_X^{[1,2>}[[z]]z^{-1}$ Then:*

(A) The map (5.14) is a vector space isomorphism.

(B) If $N \subset M$ is a non-zero submodule, then $N \cap M_0$ is also non-zero.

Proof of Lemma 5.3. (A) Map (5.14) is surjective by the assumption. To prove injectivity, extend $\succeq$ to the lexicographic order on the set of monomials $x^1x^2 \ldots x^k \otimes m$. Let $\gamma \in \mathrm{Ker}\Psi$ and $\gamma = \gamma_0 + \cdots$, where γ_0 is the leading (w.r.t. the lexicographic ordering) non-zero term. Write $\gamma_0 = x^1x^2 \ldots x^k \otimes m$. Then

$$y^1 y^2 \cdots y^k \Psi(\gamma) = 0,$$

where we choose y^j to be $\frac{1}{n}\partial_{i,n}$ if $x^j = \Omega^i_{-n}$ or $\frac{1}{n}\omega_{i,n}$ if $x^j = D^i_{-n}$. The relations of Lemma 4.3 imply that $[\partial_{i,n}, \omega_{j,-m}] = n\delta_{ij}\delta_{nm}$, and so thanks to (5.1)

$$y^1 y^2 \cdots y^k \Psi(\gamma) = \frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2} \cdots \frac{\partial}{\partial x^k}(x^1 x^2 \cdots x^k)\Psi(m).$$

Therefore $\Psi(m) = 0$, but the restriction of Ψ to M_0 being the identity, m has to be zero, hence $\gamma = 0$, as desired.

Proof of item (B) is very similar: one has to pick a non-zero $\gamma \in N$ of the lowest degree, and then apply to the highest degree term, γ_0, and appropriate $y^1 y^2 \cdots y^k$ so as to produce a non-zero element of $N \cap M_0$. $\square$

Theorem 5.2 follows from Lemma 5.3 easily. We have the adjunction morphism

$$\mathcal{Z}hu_\chi \circ \Phi \to \mathrm{Id}_{Mod_\chi - \mathcal{D}_X^{\mathrm{ch,tw}}}, \tag{5.15}$$

hence

$$\mathcal{Z}hu_\chi \circ \Phi(\mathcal{M}) \to \mathcal{M}. \tag{5.16}$$

for each $\mathcal{D}_X^{ch,tw}$-module $\mathcal{M}$. The restriction of (5.16) to $\mathcal{M}_0$ is the identity. By construction, sect. 3.5.1, $\mathcal{Z}hu_\chi \circ \Phi(\mathcal{M})$ is generated by $\mathcal{M}_0 = \Phi(\mathcal{M})$. Therefore, due to Lemma 5.3, it is an isomorphism, hence (5.15) is a functor isomorphism. This proves (5.13).

Φ is exact because it is an equivalence of categories; alternatively, the exactness follows, immediately, from Lemma 5.3. The proof of Theorem 5.2 (1, 2) is completed. The locally trivial case, i.e., assertion (3) is proved in the same way. $\square$

Remark 5.4. The condition that each $\mathcal{M}$ be generated by $\mathcal{M}_0 \subset \mathcal{M}$ in the definition of a $\mathcal{D}_X^{ch,tw}$-module looks unnecessarily restrictive. Indeed, one can do without it at least when $\mathcal{D}_X^{ch,tw}$ is 'nice.'

There is an obvious version of the definition of a $\mathcal{D}_X^{ch,tw}$-module, where the generation by $\mathcal{M}_0 \subset \mathcal{M}$ is replaced with the requirement that filtration (5.4) exist. Call a $\mathcal{D}_X^{ch,tw}$ *locally trivial* if locally on X there is an abelian basis $\tau^{(1)}, \tau^{(2)}, \ldots \subset \mathcal{T}_X$ and its lift to $\hat{\tau}^{(1)}, \hat{\tau}^{(2)}, \ldots \subset (\mathcal{D}_X^{ch,tw})_1$ so that $\hat{\tau}^{(i)}_{(n)} \hat{\tau}^{(j)} = 0$ for all i, j and $n \geq 0$. One can show the following version of Theorem 5.2 is valid for a locally trivial $\mathcal{D}_X^{ch,tw}$:

the functor

$$\mathrm{Sing}:\ Mod_\chi - \mathcal{D}_X^{ch,tw} \to Mod - \mathcal{D}_X^\chi$$

$$\mathcal{M} \mapsto \mathrm{Sing}\mathcal{M} \stackrel{\mathrm{def}}{=} \{m \in \mathcal{M}:\ a_n m = 0 \text{ for all } a \in \mathcal{D}_X^{ch,tw}, n > 0\}$$

is an equivalence of categories. We are planning to return to this topic in a susequent publication.

6. Example: chiral modules over flag manifolds

6.1. Sheaf cohomology realization of various $\hat{\mathfrak{g}}$-modules.

6.1.1. *Bernstein-Beilinson localization.* Let G be a complex simple Lie group, $B, B_- \subset G$ a generic pair of Borel subgroups, $\mathfrak{g} = \mathrm{Lie}\, G$, and $X = G/B_-$, the flag manifold. Consider the Beilinson-Bernstein [BB1] localization functor

$$\Delta : Mod_{ch(\lambda)} - \mathfrak{g} \to Mod - \mathcal{D}_X^\lambda, \tag{6.1}$$

where we regard $\lambda \in H^1(X, \Omega_X^1 \to \Omega_X^{2,cl})$ as a weight, i.e., an element of the dual to a Cartan subalgebra of $\mathfrak{g}$, cf. sect. 4.3.3, especially (4.20), and $Mod_{ch(\lambda)} - \mathfrak{g}$ is the full subcategory of the category of $\mathfrak{g}$-modules with central character $ch(\lambda)$; the latter is determined naturally by λ and assigns to a central element the number by which it acts on a module with highest weight λ. Functor (6.1) is an equivalence of categories if λ is dominant regular [BB2].

To see some examples, denote by V_λ the simple finite dimensional $\mathfrak{g}$-module with highest weight λ, M_λ the Verma module with highest weight λ, M^c_λ the corresponding contragredient Verma module. We have

$$\Delta(V_\lambda) = \mathcal{O}(\lambda), \tag{6.2}$$

$$\Delta(M^c_\lambda) = i_* i^* \mathcal{O}(\lambda), \tag{6.3}$$

where $\mathcal{O}(\lambda)$ is the sheaf of sections of the line bundle $G \times_{B_-} \mathbb{C}$, $X_e \stackrel{\text{def}}{=} \bar{B} \subset X$ is the big cell, $i : X_e \hookrightarrow X$.

6.1.2. *Chiralization.* Recall that each TCDO on X is locally trivial, see (4.20). Having fixed $\chi = \chi(z) \in \mathfrak{h}((z))$ with regular singularity, we obtain the functor

$$\mathcal{Z}hu_\chi \circ \Delta : Mod_{ch(\chi_0)} - \mathfrak{g} \to Mod_\chi - \mathcal{D}^{ch,tw}_X, \chi_0 = \mathrm{res}_{z=0}\chi(z), \tag{6.4}$$

which is an equivalence of categories if χ_0 is dominant regular ([BB1, BB2] and Theorem 5.2.)

According to Lemma 4.6, there is a vertex algebra morphism

$$\rho : V_{-h^\vee}(\mathfrak{g}) \to \Gamma(X, \mathcal{D}^{ch,tw}_X) \text{ s.t. } \rho(\mathfrak{z}(V_{-h^\vee}(\mathfrak{g})) \subset H_X. \tag{6.5}$$

Hence $\mathcal{Z}hu_\chi \circ \Delta(M)$ is a sheaf of $V_{-h^\vee}(\mathfrak{g})$-modules with central character $\chi \circ \rho$, where χ is understood as in (5.5). In particular, $\Gamma(X_e, \mathcal{Z}hu_\chi \circ \Delta(M^c_{\chi_0}))$ is a *Wakimoto module of critical level* [W, FF1, F1]. Indeed, according to (6.3), $\Gamma(X_e, \Delta(M^c_{\chi_0}))$ is but the space of functions on the big cell X_e carrying an action of $\mathfrak{g}$ twisted by λ; the definition of the functor $\mathcal{Z}hu$ in these circumstances simply mimics the Feigin-Frenkel definition of the Wakimoto module of critical level with highest weight χ_0.

Since $\Gamma(X, \Delta(M^c_{\chi_0})) = M^c_{\chi_0}$, we see that the space of global sections $\Gamma(X, \mathcal{Z}hu_\chi \circ \Delta(M^c_{\chi_0}))$ is the same Wakimoto module of critical level.

It follows from (6.2, 6.3) that $\Gamma(X_e, \Delta(V_{\chi_0})) = M^c_{\chi_0}$, and so $\Gamma(X_e, \mathcal{Z}hu_\chi \circ \Delta(V_{\chi_0}))$ is also a Wakimoto module of critical level. What can we say about its space of global sections?

It is easy to see [MS] that

$$\Gamma(X, \mathcal{M}) \text{ is the maximal } \mathfrak{g} - \text{integrable submodule of } \Gamma(X_e, \mathcal{M}) \tag{6.6}$$

Conjecturally, the maximal $\mathfrak{g}$-integrable submodule of a Wakimoto module of critical level – and arbitrary highest weight – is an irreducible $\widehat{\mathfrak{g}}$-module; therefore $\Gamma(X, \mathcal{Z}hu_\chi \circ \Delta(V_{\chi_0}))$ is also expected to be $\widehat{\mathfrak{g}}$-irreducible. We will see how this comes about in the case where either χ_0 is a regular dominant highest weight – as has been assumed so far – or $\mathfrak{g} = sl_2$ and $\chi(z) = \chi_0/z$, where χ_0 is an arbitrary integer.

Continuing under the assumption that χ_0 is a regular dominant integral highest weight we obtain a map

$$\Delta(V_{\chi_0}) \to \mathcal{Z}hu_\chi \circ \Delta(V_{\chi_0}), \tag{6.7}$$

hence a map

$$V_{\chi_0} \to \Gamma(X, \mathcal{Z}hu_\chi \circ \Delta(V_{\chi_0})), \tag{6.8}$$

Introduce the *Weyl module of critical level* $\mathbb{V}_\lambda = \mathrm{Ind}^{\widehat{\mathfrak{g}}}_{\widehat{\mathfrak{g}}_\leq} V_\lambda$, where $\mathfrak{g}_\leq$operates on V_λ via the evaluation map $\mathfrak{g}_\leq \to \mathfrak{g}$, and $K \mapsto -h^\vee$, cf. sect. 2.2.1. Note that $\mathbb{V}_0$ is nothing but the vertex algebra $V_{-h^\vee}(\mathfrak{g})$.

The universality property of induced modules implies that (6.8) uniquely extends to a $\hat{\mathfrak{g}}$-morphism

$$\mathbb{V}_{\chi_0} \to \Gamma(X, \mathcal{Z}hu_\chi \circ \Delta(V_{\chi_0})). \tag{6.9}$$

This map has kernel, because $\mathbb{V}_\lambda$ carries an action of the center, $\mathfrak{z}(V_{-h^\vee}(\mathfrak{g}))$, see Lemma 4.6. Define the *restricted Weyl module of central character* $\chi(z)$ to be

$$\mathbb{V}_{\chi(z)} = \mathbb{V}_{\chi_0}/\{(p_{(n)} - \chi(\rho(p))_{(n)})v,\ p \in \mathfrak{z}(V_{-h^\vee}(\mathfrak{g})), v \in \mathbb{V}_{\chi_0}.\} \tag{6.10}$$

Then (6.9) factors through

$$\mathbb{V}_{\chi(z)} \to \Gamma(X, \mathcal{Z}hu_\chi \circ \Delta(V_{\chi_0})). \tag{6.11}$$

Frenkel and Gaitsgory [FG3] have proved that $\mathbb{V}_{\chi(z)}$ is an irreducible $\hat{\mathfrak{g}}$-module.

Theorem 6.1. *If χ_0 is regular dominant, then map (6.11) is an isomorphism. In particular, $\Gamma(X, \mathcal{Z}hu_\chi \circ \Delta(V_{\chi_0}))$ is an irreducible $\hat{\mathfrak{g}}$-module.*

Before we continue, let us note that for any smooth variety X, even though $\overset{\circ}{\mathcal{D}}{}_X^{ch,tw}$ is graded, objects of $Mod_\chi - \overset{\circ}{\mathcal{D}}{}_X^{ch,tw}$ tend to be only filtered, because quotienting out by the character $\chi(z)$ does not respect the grading – except when

$$\chi(z) = \frac{\chi_0}{z},\ \chi_0 \in H^1(X, \Omega_X^{1,cl}). \tag{6.12}$$

If χ_0 is integral and $\mathcal{L}$ is the invertible sheaf of $\mathcal{O}_X$-modules with Chern class represented by χ_0, then Theorem 5.2 reads: $Mod_\chi - \overset{\circ}{\mathcal{D}}{}_X^{ch,tw}$ is equivalent to $Mod - \mathcal{D}_X^{\mathcal{L}}$, where $\mathcal{D}_X^{\mathcal{L}}$ is the algebra of differential operators acting on $\mathcal{L}$. In particular, associated to $\mathcal{L} \in Mod - \mathcal{D}_X^{\mathcal{L}}$ is $\mathcal{Z}hu_\chi(\mathcal{L}) \in Mod_\chi - \overset{\circ}{\mathcal{D}}{}_X^{ch,tw}$. The grading of $\overset{\circ}{\mathcal{D}}{}_X^{ch,tw}$ induces that of $\mathcal{Z}hu_\chi(\mathcal{L})$:

$$\mathcal{Z}hu_\chi(\mathcal{L}) = \mathcal{Z}hu_\chi(\mathcal{L})_0 \oplus \mathcal{Z}hu_\chi(\mathcal{L})_1 \oplus \cdots \text{ where } \mathcal{Z}hu_\chi(\mathcal{L})_0 = \mathcal{L}.$$

Denote

$$\mathcal{L}^{ch} = \mathcal{Z}hu_\chi(\mathcal{L})$$

and think of it as *chiralization* of $\mathcal{L}$.

Suppose now $\mathfrak{g} = sl_2$; then $G/B = \mathbb{P}^1$, $H^1(X, \Omega_X^{1,cl}) = \mathbb{C}$, and we let $\chi(z) = n/z$, $n \in \mathbb{Z}$. We have $\Delta(V_n) = \mathcal{O}(n)$ if $n \geq 0$ and, independently of the sign of n, $\mathcal{O}(n)$ is a $\mathcal{D}^n_{\mathbb{P}^1}$-module. Therefore, in accordance with the remark above, we denote by $\mathcal{O}(n)^{ch}$ the sheaf $\mathcal{Z}hu_{n/z}(\mathcal{O}(n))$. The sheaf $\mathcal{O}(0)^{ch}$ was one of the first examples of a CDO, and it appeared in [MSV] under the name of the *chiral structure sheaf*.

In this situation, Theorem 6.1 specializes and extends as follows:

Theorem 6.2. *Let $\mathbb{L}_n$ be the unique irreducible highest weight module over $\widehat{sl}_2$ at the critical level with highest weight n. Then*

(i) If $n \in \{0, 1, 2, ...\}$, then there are $\widehat{sl_2}$-module isomorphisms

$$H^0(\mathbb{P}^1, \mathcal{O}(n)^{ch}) \xrightarrow{\sim} H^1(\mathbb{P}^1, \mathcal{O}(n)^{ch}) \xrightarrow{\sim} \mathbb{L}_n.$$

(ii) If $n \in \{-2, -3, -4, ...\}$, then there are $\widehat{sl_2}$-module isomorphisms

$$H^0(\mathbb{P}^1, \mathcal{O}(n)^{ch}) \xrightarrow{\sim} H^1(\mathbb{P}^1, \mathcal{O}(n)^{ch}) \xrightarrow{\sim} \mathbb{L}_{-n-2}.$$

(iii) If $n = -1$, then

$$H^0(\mathbb{P}^1, \mathcal{O}(-1)^{ch}) = H^1(\mathbb{P}^1, \mathcal{O}(-1)^{ch}) = 0.$$

6.2. **Proofs.**

6.2.1. *Proof of Theorem 6.1.* Our discussion will heavily rely on results of [FG1, FG2, FG3]. Denote by $\mathcal{O}^{\rm crit}$ the version of the $\mathcal{O}$-category of $\hat{\mathfrak{g}}$ modules at the critical level, where all modules A are assumed to be filtered, $A=\cup_{i=-\infty}^{+\infty}F_iA$, in such a way that $F_iA=0$ if $i<<0$ and $\mathfrak{g}\otimes t^j(F_iA)\subset F_{i-j}A$.

Denote by $\mathfrak{z}$ the center of the completed universal enveloping algebra of $\widehat{\mathfrak{g}}$ at the critical level [FF2]. Any object of $\mathcal{O}^{\rm crit}$ is a $\mathfrak{z}$-module. Denote by $\mathcal{O}^{\rm crit}_{\lambda(z)}$ the full subcategory of $\mathcal{O}^{\rm crit}$ where $\mathfrak{z}$ acts according to the character $\lambda(z)$.

Finally, let $\mathcal{O}^{{\rm crit},G}_{\lambda(z)}$ be the full subcategory of $\mathcal{O}^{\rm crit}_{\lambda(z)}$ consisting of $\mathfrak{g}$-integrable modules. Note that, by definition, $\mathbb{V}_{\chi(z)}\in{\rm Ob}\,\mathcal{O}^{{\rm crit},G}_{\lambda(z)}$ provided $\chi(z)$ and $\lambda(z)$ match, i.e., $\lambda=\chi\circ\rho$, see (6.5). Likewise, $\Gamma(X,\mathcal{Z}hu_\chi\circ\Delta(V_{\chi_0}))\in{\rm Ob}\,\mathcal{O}^{{\rm crit},G}_{\lambda(z)}$ thanks to (6.6). It is a fundamental result of Frenkel and Gaitsgory [FG3] that $\mathcal{O}^{{\rm crit},G}_{\lambda(z)}$ is semi-simple, and $\mathbb{V}_{\chi(z)}$ is its unique irreducible object. This implies that map (6.11) is injective, and it remains to prove surjectivity.

An embedding $\mathbb{V}_{\chi(z)}\to A$, $A\in{\rm Ob}\,\mathcal{O}^{{\rm crit},G}_{\lambda(z)}$ is determined by a *singular vector of weight* χ_0, i.e., $v\in A$ such that (1) v is annihilated by $\mathfrak{g}[t]t\oplus\mathfrak{n}_+$, and (2) $\mathfrak{h}\subset\mathfrak{g}$ operates on $\mathbb{C}v$ according to χ_0. On the other hand, the semi-simplicity of $\mathcal{O}^{{\rm crit},G}_{\lambda(z)}$ implies that $\Gamma(X,\mathcal{Z}hu_\chi\circ\Delta(V_{\chi_0}))$ is a direct sum of copies of $\mathbb{V}_{\chi(z)}$. Hence it remains to show that there is a unique up to proportionality singular vector of weight χ_0 in $\Gamma(X,\mathcal{Z}hu_\chi\circ\Delta(V_{\chi_0}))$. In fact, more is true: the entire $\Gamma(X_e,\mathcal{Z}hu_\chi\circ\Delta(V_{\chi_0}))$ contains only one up to proportionality singular vector of weight χ_0.

To see this, recall that the Wakimoto module $\Gamma(X_e,\mathcal{Z}hu_\chi\circ\Delta(V_{\chi_0}))$ is free over $\mathfrak{n}_+[t^{-1}]t^{-1}$ and co-free over $\mathfrak{n}_+[t]$ with one generator; this fact has been the cornerstone of the Wakimoto module theory since its inception in [FF1]. A little more precisely, if we let 1 be the function equal to 1 on the big cell $X_e\subset X$, then $U(\mathfrak{n}_+[t^{-1}]t^{-1})1\subset\Gamma(X_e,\mathcal{Z}hu_\chi\circ\Delta(V_{\chi_0}))$ is free and there is an $\mathfrak{n}_+[t,t^{-1}]$ module isomorphism

$$\Gamma(X_e,\mathcal{Z}hu_\chi\circ\Delta(V_{\chi_0}))\stackrel{\sim}{\to}Hom_{\mathfrak{n}_+[t^{-1}]t^{-1}}(U(\mathfrak{n}_+[t,t^{-1}]),U(\mathfrak{n}_+[t^{-1}]t^{-1})1),$$

where Hom is meant to be the *restricted Hom*, which is defined to be a direct sum of weight components

$$\oplus_{\alpha,\beta}Hom_{\mathfrak{n}_+[t^{-1}]t^{-1}}(U(\mathfrak{n}_+[t,t^{-1}])_\alpha,U(\mathfrak{n}_+[t^{-1}]t^{-1})_\beta 1),$$

α and β varying over the semi-lattice spanned by positive roots of $\mathfrak{g}$.

It follows that for any $x\in\Gamma(X_e,\mathcal{Z}hu_\chi\circ\Delta(V_{\chi_0}))$ there is a $u\in U(\mathfrak{n}_+[t])$ so that $0\neq ux\in U(\mathfrak{n}_+[t^{-1}]t^{-1})1$. Therefore, singular vectors may occur only in $U(\mathfrak{n}_+[t^{-1}]t^{-1})1$. Weight space decomposition of the latter is given by

$$\begin{aligned}U(\mathfrak{n}_+[t^{-1}]t^{-1})1&=\oplus_\alpha(U(\mathfrak{n}_+[t^{-1}]t^{-1})1)_{\chi_0+\alpha},\\(U(\mathfrak{n}_+[t^{-1}]t^{-1})1)_{\chi_0+\alpha}&=U(\mathfrak{n}_+[t^{-1}]t^{-1})_\alpha 1.\end{aligned}$$

Therefore, $(U(\mathfrak{n}_+[t^{-1}]t^{-1})1)_{\chi_0}$ is one-dimensional and spanned by 1, a unique up to proportionality singular vector of weight χ_0. $\square$

6.2.2. *Proof of Theorem 6.2*. Of course item (i) is a particular case of Theorem 6.1, but items (ii, iii) are not. For the reader's convenience we will give an independent proof of all three items based on representation theory of $\widehat{sl}_2$ as developed in [M], where information more complete than in the general case is available; an alternative approach would be to use [FF1].

Let $\mathbb{M}_\nu$ be the Verma module over $\widehat{sl}_2$ at the critical level; this means that $\mathbb{M}_\nu$ is a universal highest weight module, where the highest weight vector v satisfies $h_0 v = \nu v; Kv = -2v$, cf. sect. 2.2.1; we will also be using some explicit formulas from sect. 4.3.3.

$\mathbb{M}_\nu$ has a unique non-trivial maximal submodule; denote by $\mathbb{L}_\nu$ the corresponding irreducible quotient.

The Verma module $\mathbb{M}_\nu$ is always reducible, because the Sugawara operators, which in the vertex algebra notation become $T_n = (e_{-1}f + f_{-1}e + 1/2h_{-1}h)_n$, commute with the action of $\widehat{sl}_2$. Define the quotient

$$\mathbb{M}_{\nu/z} = M_\nu / \sum_{n>0} T_{-n}(\mathbb{M}_\nu).$$

The module $\mathbb{M}_{\nu/z}$ is irreducible unless $\nu \in \mathbb{Z} - \{-1\}$. If $\nu = n \in \mathbb{Z} - \{-1\}$, then $\mathbb{M}_{\nu/z}$ is reducible and contains a unique non-trivial submodule isomorphic to $\mathbb{L}_{-n-2}$. We obtain the following exact sequence

$$0 \to \mathbb{L}_{-n-2} \to \mathbb{M}_{n/z} \to \mathbb{L}_n \to 0. \tag{6.13}$$

The difference between n positive and negative lies in that if $n \geq 0$, then $\mathbb{L}_{-n-2}$ is generated, as a submodule, by f_0^{n+1} applied to the highest weight vector of $\mathbb{M}_{n/z}$; therefore, $\mathbb{L}_n$ is sl_2-integrable. On the other hand, if $n < -1$, then $\mathbb{L}_{-n-2}$ is generated by e_{-1}^{-n-1} applied to the highest weight vector of $\mathbb{M}_{n/z}$; therefore, $\mathbb{L}_n$ is not sl_2-integrable, but then $\mathbb{L}_{-n-2}$ is.

Let us now prove the assertions about the space of global sections in (i, ii,iii). In order to compute $H^0(\mathbb{P}^1, \mathcal{O}(n)^{ch})$, we observe that there is a map

$$\mathbb{M}_n \to \Gamma(\mathbb{C}_0, \mathcal{O}(n)^{ch}) \tag{6.14}$$

that sends the highest weight vector $v \in \mathbb{M}_n$ to $1 \in \Gamma(\mathbb{C}_0, \mathcal{O}(n)^{ch})$, also a highest weight vector.

If $n \geq 0$, $1 \in \Gamma(\mathbb{C}_0, \mathcal{O}(n)^{ch})$ is annihilated by f_0^{n+1} (because $1 \in \Gamma(\mathbb{P}^1, \mathcal{O}(n))$, and $\Gamma(\mathbb{P}^1, \mathcal{O}(n))$ is the $(n+1)$-dimensional irreducible sl_2-module.) Therefore, (6.14) factors through the map

$$\mathbb{L}_n \to \Gamma(\mathbb{C}_0, \mathcal{O}(n)^{ch}). \tag{6.15}$$

Since $\Gamma(\mathbb{C}_0, \mathcal{O}(n)^{ch})$ has the same character as $\mathbb{M}_n$, this implies that $\Gamma(\mathbb{C}_0, \mathcal{O}(n)^{ch})$ fits into the following exact sequence

$$0 \to \mathbb{L}_n \to \Gamma(\mathbb{C}_0, \mathcal{O}(n)^{ch}) \to \mathbb{L}_{-n-2} \to 0. \tag{6.16}$$

Therefore $\mathbb{L}_n$ is its unique non-trivial, hence maximal, submodule, which is sl_2-integrable, as it was explained above. Now (6.6) implies an isomorphism $H^0(\mathbb{P}^1, \mathcal{O}(n)^{ch}) \xrightarrow{\sim} \mathbb{L}_n$.

In the case where $n < -1$, map (6.14) is an isomorphism, because the unique non-trivial submodule $\mathbb{M}_n$ is generated by $e_{-1}^{-n-1}v$, and map (6.14) sends the latter to $(\partial_x)_{-1}^{-n-1}1 \neq 0$, as formula (4.17) implies. The brief discussion after (6.13) shows

that if $n < -1$, then the maximal integrable submodule is $\mathbb{L}_{-n-2}$ and so is the space of global sections.

Finally, $\Gamma(\mathbb{C}_0, \mathcal{O}(-1)^{ch})$ is irreducible and not integrable, and so the space of global sections is zero.

It remains to show that in each of three cases $H^1(\mathbb{P}^1, \mathcal{O}(n)^{ch})$ is isomorphic to $H^0(\mathbb{P}^1, \mathcal{O}(n)^{ch})$. We will achieve that by computing the Euler character of $\mathcal{O}(n)^{ch}$ in two different ways.

Since $\mathcal{O}(n)^{ch} = \oplus_{j\geq 0}\mathcal{O}(n)^{ch}_j$, we can introduce the Euler characteristic $\mathrm{Eu}(\mathcal{O}(n)^{ch}_j) = \dim H^0(\mathbb{P}^1, \mathcal{O}(n)^{ch}_j) - \dim H^1(\mathbb{P}^1, \mathcal{O}(n)^{ch}_j)$ and the Euler character

$$\mathrm{Eu}(\mathcal{O}(n)^{ch})(q) = \sum_{j=0}^{\infty} q^j \mathrm{Eu}(\mathcal{O}(n)^{ch}_j).$$

On the other hand, we can similarly consider the formal characters $ch(H^i(\mathbb{P}^1, \mathcal{O}(n)^{ch}))(q) = \sum_{j\geq 0} q^j \dim H^i(\mathbb{P}^1, \mathcal{O}(n)^{ch}_j)$, $i = 0, 1$, and obtain

$$\mathrm{Eu}(\mathcal{O}(n)^{ch})(q) = ch(H^0(\mathbb{P}^1, \mathcal{O}(n)^{ch}))(q) - ch(H^1(\mathbb{P}^1, \mathcal{O}(n)^{ch}))(q). \tag{6.17}$$

The characters of irreducible $\widehat{sl}_2$-modules at the critical level have been known since [M]; for example,

$$ch(H^0(\mathbb{P}^1, \mathcal{O}(n)^{ch}))(q) = ch\,\mathbb{L}_n = \frac{n+1}{1-q^{n+1}}\prod_{j=1}^{\infty}(1-q^j)^{-2} \text{ if } n \geq 0. \tag{6.18}$$

On the other hand, the Euler character $\mathrm{Eu}(\mathcal{O}(n)^{ch})(q)$ can be computed independently. The sheaf $\mathcal{O}(n)^{ch}$ carries a filtration such that the associated graded object is a direct sum of sheaves $\mathcal{O}(2s+n)$, $s \in \mathbb{Z}$; this is what (3.6) amounts to in this case. Therefore, we can as well compute the Euler character of the associated graded object. This is as follows:

Informally speaking (cf. [MSV], sect. 5.8), the local section $(\partial_x)_{-s_1}\cdots(\partial_x)_{-s_p}x_{-t_1}\cdots x_{-t_q}$ contributes to the graded object the sheaf $\mathcal{O}(2p - 2q+n)$ sitting in conformal weight $\sum_i(s_j+t_j)$-component. Since $\mathrm{Eu}\ \mathcal{O}(2p-2q+n) = 2p-2q+n+1$, hence $\mathrm{Eu}\ \mathcal{O}(2p-2q+n) + \mathrm{Eu}\ \mathcal{O}(2q-2p+n) = 2(n+1)$, the Euler character of $\mathcal{O}(n)^{ch}_j$ equals the number of 2-colored partitions of j times $(n+1)$. We obtain then

$$\mathrm{Eu}(\mathcal{O}(n)^{ch})(q) = (n+1)\prod_{j=1}^{\infty}(1-q^j)^{-2} \tag{6.19}$$

Plugging this and (6.18) in (6.17) gives us

$$ch(H^1(\mathbb{P}^1, \mathcal{O}(n)^{ch}))(q) = q^{n+1}ch\mathbb{L}_n = \frac{(n+1)q^{n+1}}{1-q^{n+1}}\prod_{j=1}^{\infty}(1-q^j)^{-2} \text{ if } n \geq 0. \tag{6.20}$$

We see that $ch(H^1(\mathbb{P}^1, \mathcal{O}(n)^{ch}))(q)$ equals $ch(H^0(\mathbb{P}^1, \mathcal{O}(n)^{ch}))(q)$ up to an overall power of q. Since an irreducible module is determined by its character, we conclude that $H^1(\mathbb{P}^1, \mathcal{O}(n)^{ch}) \xrightarrow{\sim} H^0(\mathbb{P}^1, \mathcal{O}(n)^{ch})$, as desired. Note that the shift by the factor of q^{n+1} means that $H^1(\mathbb{P}^1, \mathcal{O}(n)^{ch})$ 'grows' from the conformal weight $(n+1)$ component, unlike $H^0(\mathbb{P}^1, \mathcal{O}(n)^{ch})$, which grows from the conformal weight zero component.

The case of $n < -1$ is handled similarly; an untiring reader will discover that in this case it is $H^1(\mathbb{P}^1, \mathcal{O}(n)^{ch})$ that grows from the conformal weight zero component, while $H^0(\mathbb{P}^1, \mathcal{O}(n)^{ch})$ originates in the conformal weight $(-n-1)$ component.

The case where $n = -1$ all the characters in sight are obviously equal to zero. Theorem 6.2 is now proved. $\square$

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T.A.: Department of Mathematics, Nara Women's University, Nara 630-8506, Japan. E-mail address: arakawa@cc.nara-wu.ac.jp

D.Ch.: Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA. E-mail:chebotar@usc.edu

F.M.: Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA. E-mail:fmalikov@usc.edu